Spherical geometry

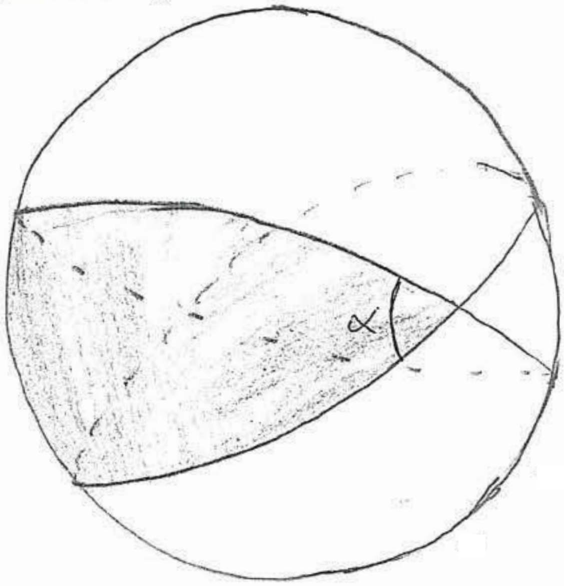
Although Euclidean geometry has been the most accepted and the most widely used type of geometry for centuries, mathematicians and geometers have discovered other types of geometry that can be just as useful. These geometries are referred to as non Euclidean geometries. There are two types of non Euclidean geometries: Spherical and hyperbolic geometry. In this exploration. I shall focus on spherical geometry. This is the geometry on the surface of spheres discovered by Riemann in 1854. This geometry agrees with four of Euclidean geometry postulates except the parallel lines postulate. In Riemannian geometry, there are no parallel lines (All lines meet in spherical geometry). In this type of geometry, it is impossible to draw straight lines because as soon as you start drawing a straight line, it curves according to the curvature of the surface of the sphere.

Aim and rationale not clearly stated.

In spherical geometry, all straight lines form great circles. This can be illustrated if you attempt to draw a straight line on the surface of a football. Eventually, the marker will end up where you started. You have, in effect, drawn a great circle. A great circle is a circle whose center is the center of the sphere. It therefore has the same radius and diameter as the sphere. In the example that I gave, the diameter of the circle formed is the same as the diameter of the football. Any two great circles must intersect at two points on the surface of the sphere

Diangles

The intersection of two great circles gives rise to two congruent antipodal diangles. A diangle is a planar figure formed on the surface of the sphere by the intersection of two great circles. It is called a 'di'angle as it only has two angles enclosed in it.



Antipodal means on opposite sides of the sphere

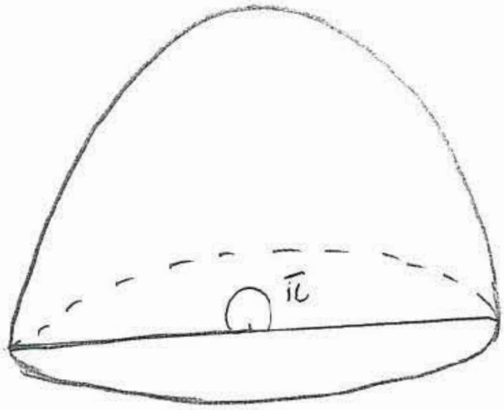
Figure 1

The two great circles enclosing the diangle make an angle α with each other. The area of the diangle is therefore given by $A=2\alpha r^2$.

Proof

Case 1: Hemisphere

A hemisphere is a diangle whose great circles make an angle $\alpha=\pi$ with each other.



Consider a hemisphere with radius 'r'

$A=\frac{4\pi r^2}{2}=2\pi r^2=2\alpha r^2$ QED

Case 2

Given a diangle with an angle α less than π, its area will be $\frac{\alpha}{\pi}$ of the area of the hemisphere i.e.

$\frac{\alpha}{\pi}\times 2\pi r^2 = 2\alpha r^2$ QED

Therefore the area of a diangle on a sphere with radius 'r' is $2\alpha r^2$

Spherical triangles

The intersection of three great circles on the surface of the sphere can form two congruent antipodal triangles. These triangles are referred to as spherical triangles. A spherical triangle is a polygon formed on the surface of a sphere whose sides are great circles. A good example is

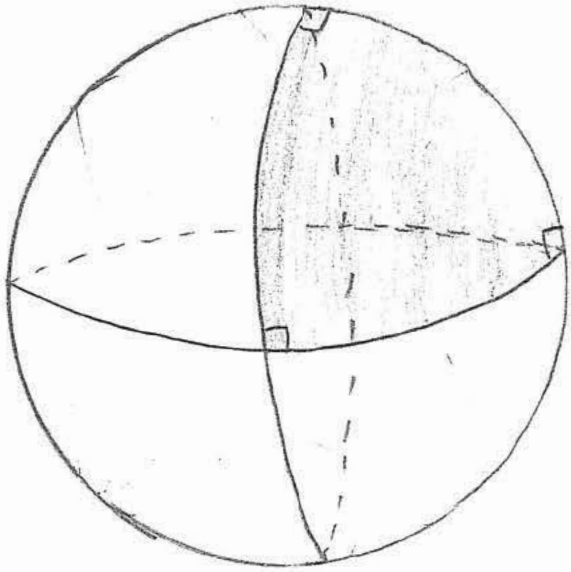
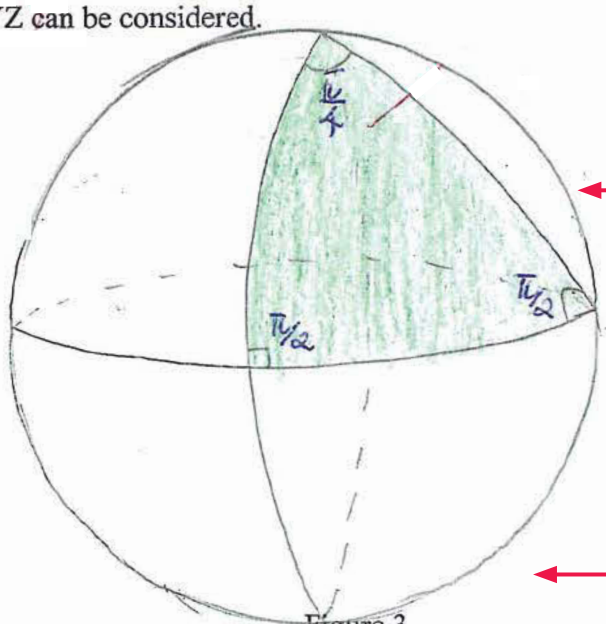


Figure 2

Triangle ABC is a spherical triangle as it has been formed on the surface of the sphere by the intersection of three great circles. I have used this triangle as a simple example to illustrate Girard's theorem (shown and proved later). However, this triangle is special as it makes 0.125 exactly of the surface of the sphere. All its interior angles are also right angles. A of this triangle is therefore $0.125(4\pi r^2)=0.5\pi r^2$.

Another triangle XYZ can be considered.



Points A,B,C need to be indicated on the diagram. Also X,Y,Z.

A Confusion between explanation and diagram.

Figure 3

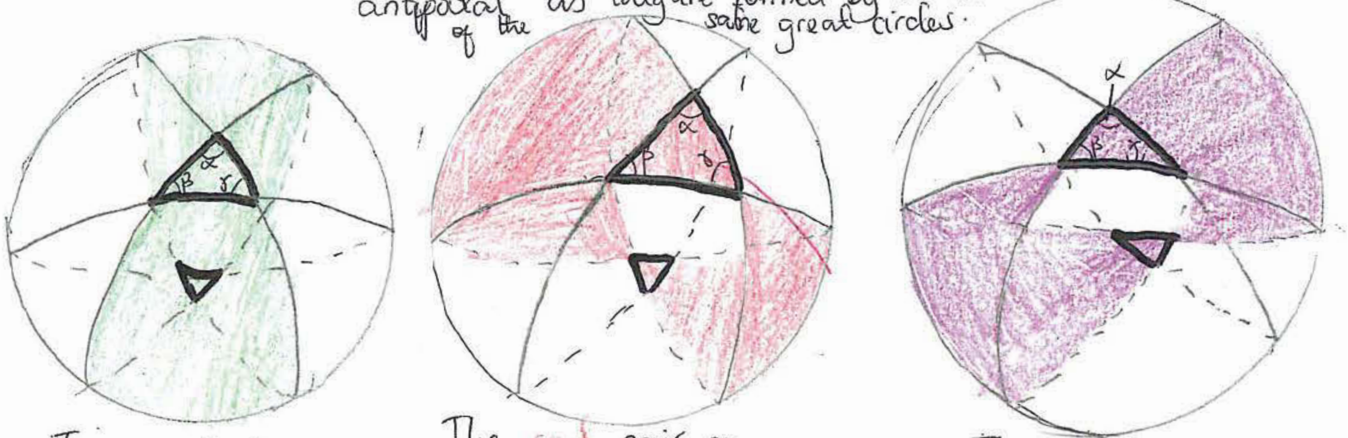
This is another special triangle that is 0.0625 of the whole surface of the sphere. Its area is therefore $A=0.0625\ (4\pi r^2)=0.25\pi r^2$

Girard's theorem

A general formula for finding the area of any given spherical triangle is $A=r^2\ (\alpha+\beta+\gamma-\pi)$ where α, β, and γ are the interior angles of the triangle and r is the radius of the sphere

Proof

The highlighted triangles are similar and antipodal as they are formed by the intersection of the same great circles.

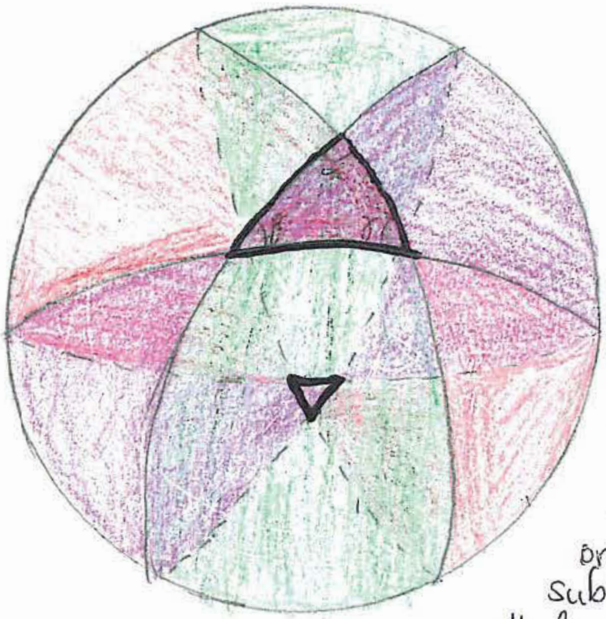


The green pair of diangles has an area of $4\alpha r^2$

The red pair of diangles has an area of $(2\gamma r^2)\cdot 2=4\gamma r^2$

The purple pair of diangles has a total area of $4\beta r^2$

This is evidence of personal engagement. Drawing these correctly is challenging.



It should be noted that each diangle pair takes into account two triangles. The total area of all diangles takes into account 6 of the triangles while there are only two on the surface of the sphere. ∴ If A is the area of one triangle, 4A has to be subtracted from the total area of all diangles.

Figure 4

Area of the sphere$=4\pi r^2$

$$=4\alpha\, r^2+4\beta\, r^2+4\gamma\, r^2-4A$$

where A is the area of one triangle.

Equating

$4\pi r^2=4\alpha\, r^2+4\beta\, r^2+4\gamma\, r^2-4A$

Therefore $A= r^2\,(\alpha+\beta+\gamma-\pi)$ QED

Applying this formula to triangle ABC (figure 2) gives $A=r^2\,(0.5\pi+0.5\,\pi+0.5\,\pi-\pi)=0.5\,\pi\, r^2$

A of triangle XYZ (figure 3)$= r^2\,(0.5\pi+0.5\,\pi+0.25\,\pi-\pi)=0.25\,\pi\, r^2$

This formula therefore is valid as gives the correct answer.

To verify this formula for more complex triangles, I constructed triangles on Geometer Sketch Pad (GSP) and measured their area (using grids). The result can then be compared by the result obtained when π is subtracted from the sum of their interior angles (Girard's theorem).

Since spherical triangles are created from the intersection of three great circles, I drew them on GSP by intersecting three circles of the same radii. I considered these to be my great circles. I measured the angles included in the triangles by drawing tangents at the triangle's vertices and measuring the angle formed by these tangents

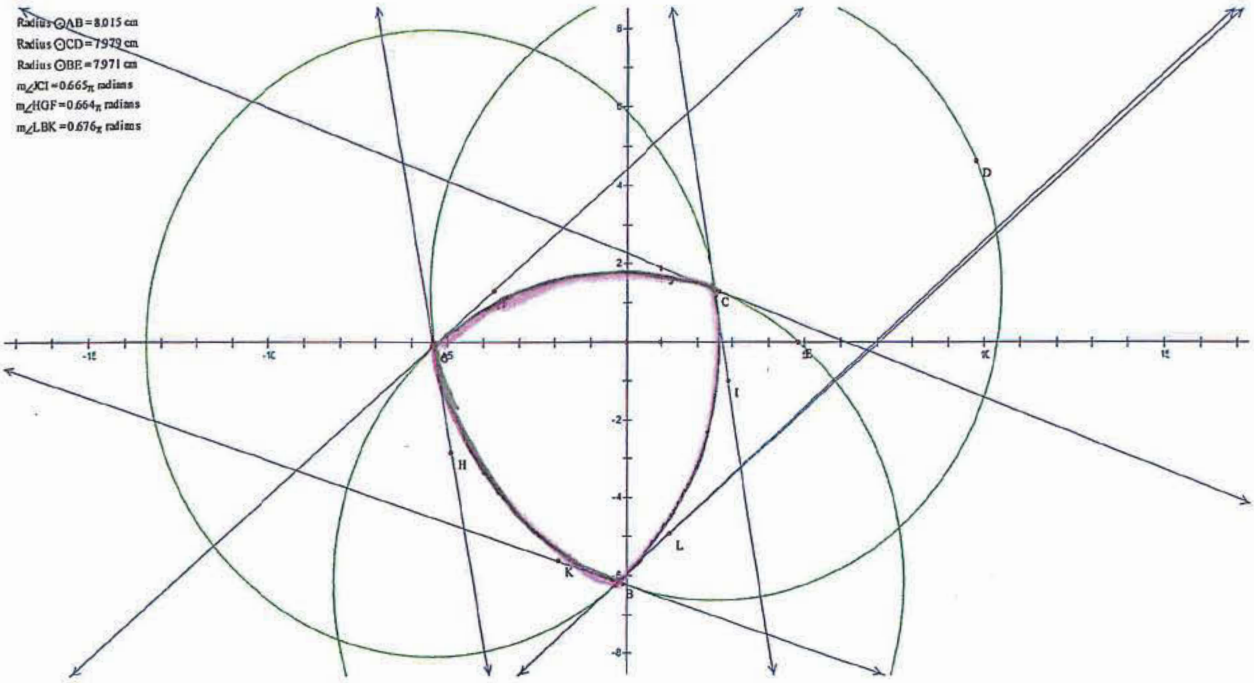


Figure 5

Each square grid is has an area of 1cm^2. However, when pasting this image onto this document, it had to be shrunk so that it could fit in this page.

These circles are of a radius of 8cm.

Using Girard's theorem,

$A = (8cm)^2(0.665 + 0.664 + 0.676 - 1)\pi$=202.067cm^2

However, this triangle on GSP only occupies about 36cm^2 because it occupies about 36 square grids in total.

I tried the same concept with a different triangle to try verify the theorem. I used circles of radi 4.5cm.

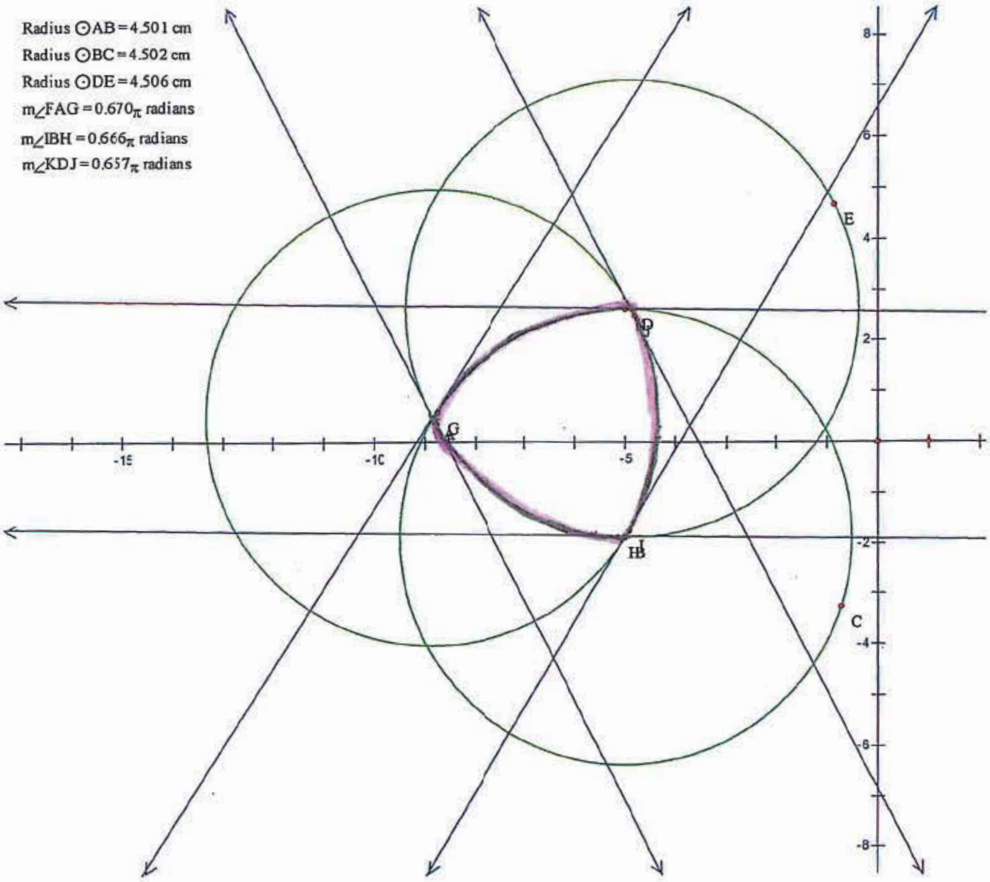


Figure 6

Each square grid is has an area of 1cm^2. However, when pasting this image onto this document, it had to be shrunk so that it could fit in this page.

These circles are of a radius of 4.5cm.

Using Girard's theorem,

$A = (4.5cm)^2(0.670 + 0.666 + 0.657 - 1)\pi$=63.172cm^2

However, this triangle on GSP only occupies about 11cm^2 because it occupies about 11 square grids on total.

Correctly noted but perhaps this is obvious.

This method of verification is therefore not valid due to the fact that it relies on the representation of spherical triangles on 2 dimensions. Spherical triangles are typically 3 dimensional figures as they are formed on the surface of the sphere. An attempt to represent them on 2D space (this page) was not successful as it lost information about the triangles. Nonetheless this proves that this is not a valid method of representing spherical triangles on 2D

Due to the fact the previous verification method failed, I shall now try another verification method to verify Girard's theorem for simple and complex triangles.

In this method, I shall use a real sphere (ball) on which I will draw a triangle. I will then draw a 1 cm grid on this triangle and count the grid squares it takes up. This will empirically give me the area of this triangle. I will then compare this value to one that I get by using Girard's theorem.

I first started with a simple triangle similar to triangle ABC (figure 2).

Clear well presented photo.

Figure 7

This triangle's interior angles are right angles only.

I counted the grids that I drew inside this triangle and found that it had included 188 grids. Since each square grid is $1cm^2$ big, this triangle therefore has an area of $188\pm2cm^2$. This value is not exact as I have left room for errors in my measurements due to the fact that the grids were drawn manually using a flexible ruler that could not curve as much as the sphere curved.

Using Girard's theorem, its area=r^2 ($0.5\pi+0.5\ \pi+0.5\ \pi-\ \pi$) =$0.5\ \pi\ r^2$ where 'r' is the radius of the ball. Due to the fact that I didn't have any other equipment apart from a flexible 30cm ruler, markers and a ball, I had to calculate the diameter hence radius of the ball instead of measuring it directly. I did this by measuring a quarter of the ball's circumference (my triangle's base). I found it to be 17.3±0.2cm. This error is due to the fact that the surface is curved and although my ruler is flexible, I could not get the exact measurement of the length. I then calculated the radius from that i.e.

$$17.3cm = \frac{\pi}{2}r \therefore r = 11.014cm$$

Consideration of error is a good example of meaningful reflection.

I propagated errors by calculating the relative error in the quarter circumference measurement and multiplying it by the result in the radius i.e.

$$\frac{0.2cm}{17.3cm} \times 11.014cm = 0.127cm$$

The value of 'r' with errors propagated into it is 11.014±0.127cm

Substituting this value for 'r' in the result I got from Girard's theorem gives

$$A = 0.5\pi(11.014cm)^2 = 190.550cm^2$$

I also propagated errors into this value of area by multiplying twice the relative error in radius by the value of area that I obtain. I am using twice the relative error in radius as it is multiplied by itself in the area equation hence introducing the same error twice

$$2\frac{0.127cm}{11.014cm} \times 190.550cm^2 = 4.394cm^2$$

The value of area of this triangle with errors propagated into it is therefore 190.550±4.394cm^2

The value 188 cm^2 that I obtained from empirically measuring the area of the triangle (using grids) is within the error band of the value I obtained from Girard's theorem (190.550±4.394cm^2). This theorem is therefore valid as it gives the same result as the empirical measurement that I did.

After proving the validity of the theorem on a simple triangle (one that only has right angles, and makes up an eighth of the surface of the sphere), I shall now move on and attempt to prove Girard's theorem on a complex triangle. On this triangle, I manually measured the included angles rather accurately. I then drew grids on the triangle and counted how many there were to get the area of the triangle.

Figure 8

I counted the grids that I drew inside this triangle and found that it had included 144 grids. Since each square grid is 1cm^2 big, this triangle therefore has an area of $147\pm2\text{cm}^2$. This value is not exact as I have left room for errors in my measurements due to the fact that the grids were drawn manually using a flexible ruler that could not curve as much as the sphere curved.

I measured the angles and found out that they are 65°, 132° and 53°. In radians these are 0.344π, 0.733π and 0.294π respectively.

Should be 0.361 not 0.344 but does not detract from flow.

Using Girard's theorem, its area=r^2 ($0.344\pi+0.733\,\pi+0.294\,\pi-\,\pi$) $=0.371\,\pi\, r^2$ where 'r' is the radius of the ball. I then used the value of 'r' that I had obtained from the previous calculation since I used the same ball for this experiment.

Substituting this value for 'r' in the result I got from Girard's theorem gives

$$A = 0.371\pi(11.014cm)^2 = 141.388cm^2$$

I also propagated errors into this value of area by multiplying twice the relative error in radius by the value of area that I obtain. I am using twice the relative error in radius as it is multiplied by itself in the area equation hence introducing the same error twice

$$2\frac{0.127cm}{11.014cm} \times 141.388cm^2 = 3.361cm^2$$

The value of area of this triangle with errors propagated into it is therefore $141.388\pm3.361\text{cm}^2$.

The value 144 cm^2 that I obtained from empirically measuring the area of the triangle (using grids) is within the error band of the value I obtained from Girard's theorem ($141.388\pm3.361\text{cm}^2$).

Error calculation evidence of meaningful reflection.

I have therefore successfully proven the validity of this theorem as it has given me the expected result in a real practical situation.

Applications of spherical geometry

Due to the fact that the Earth is almost a sphere (it is in fact oblate spheroid), spherical geometry is always used in navigation. Pilots and sailors use it all the time to find the shortest distance between their current positions and their destinations. On the Earth's surface, Euclidean geometry would not work as the Earth is curved and joining two points on it with a straight line would, in effect, mean a route through the Earth i.e. under the surface of the Earth (which is not feasible for pilots or sailors).

Spherical geometry is also very useful in architecture as buildings employing spherical geometry tend to be very attractive.

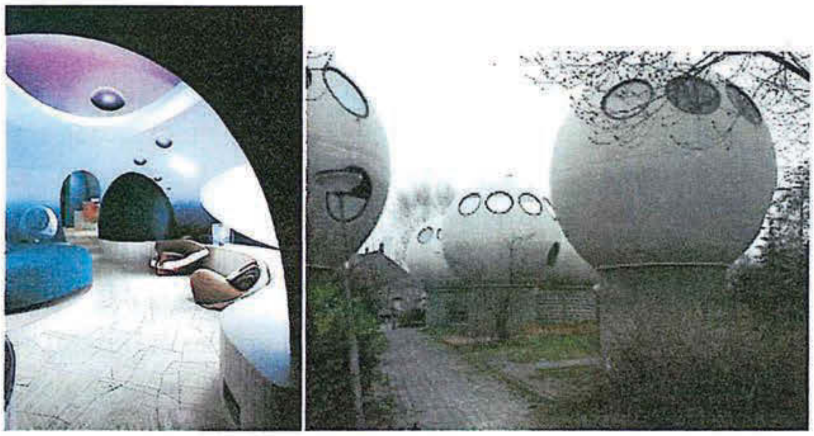


Figure 9

Furthermore, it is useful in some areas in chemistry such as nanotechnology. A good example is buckminsterfullerines which are spherical molecules. These are made of pentagons and hexagons making up a sphere. Since chemists are usually interested in finding the angle between the bonds in these bucky balls, spherical geometry might come in handy as these polygons are on a sphere therefore Riemannian geometry rules apply.

A more modern application of spherical geometry is computational origami. This is a type of computer-assisted design (CAD) program used to model the ways in which various materials, including paper, can be folded (Rouse). This has various applications in the engineering and information technology fields and could not have been easily untangled without spherical geometry. As this technology progresses, scientists will be able to manufacture items such as foldable telescopes and satellite wings (It's a small web).

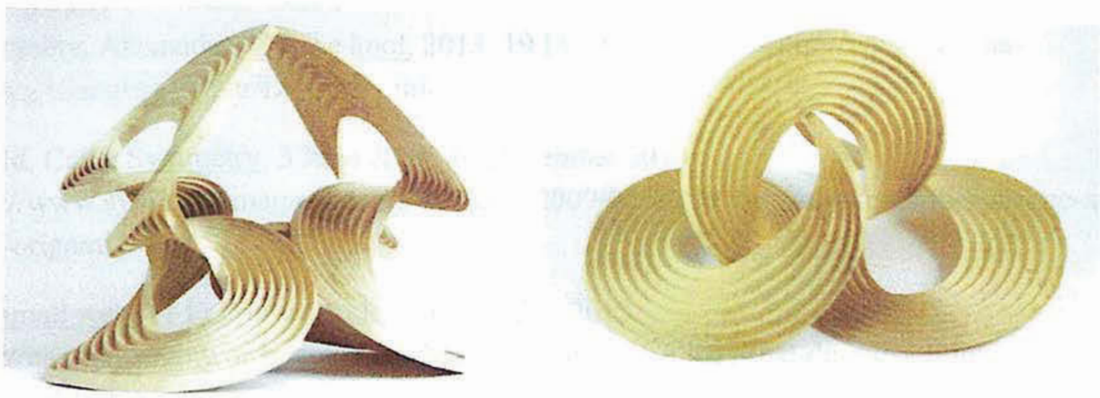


Figure 10

Reflection

I think the discovery of spherical geometry is very important as it eased navigation. I think this is very good as it made access to many parts of the world easier. This facilitates globalisation and hence development as more people have greater access to place more easily. It also enables emergencies (e.g. fire outbreaks, casualties) to be taken care of faster as transport is made faster and easier.

In addition to that, I think it is important as it has facilitated research (and maybe even breakthrough in the future!) in the frontiers of science. Concepts like computational origami would be impenetrable without the knowledge of spherical geometry. Consequently, research in these areas is promising as scientists consider the possibility of evolving information technology through storing more data in a smaller space. This would greatly influence the world and bring about even further development!

Nonetheless I believe that the discovery of spherical geometry in 1854 by Riemann is proof that our world is developing and not only remaining stagnant. This meant that there are people who are willing to research and delve into an idea deep enough to come out with a completely new phenomenon. This is very encouraging for mankind as we know that we have not lost hope; we will still trudge forward. It is encouraging as many more people will be ready to take risks and discover new items despite the controversy.

It also facilitates nanotechnology: bucky balls. If scientists can discover a way to delocallise electrons on the surface of the carbon sphere, it will be a great breakthrough in science brought about by spherical geometry.

Spherical geometry improves the beauty of our surroundings through the construction of aesthetically beautiful buildings. This brings about happiness and a less stressed society as people are surrounded by beauty.

This is reflection but not critical reflection.

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