Producing an equation to model a cooling cup of tea

Introduction

My inspiration for this investigation arose after extended amounts of time in class spent lurking in the depths of calculus. In my confusion, I wanted to apply what I'd learnt in class to something that was physical and relevant in some attempt to justify that what I was learning was *real*. At first, it was difficult to think of anywhere in day to day life that could hold some relationship to the convoluted methods of calculus, but I finally settled on something that was a fundamental aid in my Diploma. Tea.

After many revision sessions spent buried in a text book, I often found my tea cold when I got around to drinking it. Consequently I started thinking about how tea cooled, from my higher level in chemistry I knew that the difference between the cooling body and the ambient temperature (room temperature) changed the rate at which the body cooled. So that large differences between temperatures cause a large rate of change and the smaller the differences between the cooling body and the ambient temperature cause a smaller rate. From my common sense I knew that the tea couldn't go below room temperature at any point which meant that the tea would approach room temperature with the rate getting smaller and smaller and smaller as did the difference between the tea and ambient temperature. It sounded like an exponential curve, and one that could be graphed. I aimed to create a model of the rate of cooling and produce an equation that would allow me to calculate how long I could revise before my tea would be undrinkable.

My main goal in this investigation was simply apply maths to somewhere outside of the lesson environment. To be able to solve an equation so that I can produce a value that bears significance in the real world, rather than being an answer to a textbook question.

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D +

A + Aim

Gathering Data

To gather the data I needed to produce the graph I had to first measure a cup of tea cooling down. Using a temperature sensor and piece of graphing software I was able to produce a set of readings that measured the temperature of the tea to 1.d.p every 10 seconds for 2.5 hours. The graph drawn on the software reflected the exponential function that I needed, with the asymptote of the results approaching the room temperature of 24˚C

However due to practicality issues and an error in the software, only temperature readings for every 5 minutes could be recorded for later use. Although fewer results decreased the accuracy of the model the remaining readings are numerous enough that they should provide enough accuracy to form a reasonably accurate equation calculate the temperature of the tea at any given time. The readings are listed in the table on the next page.

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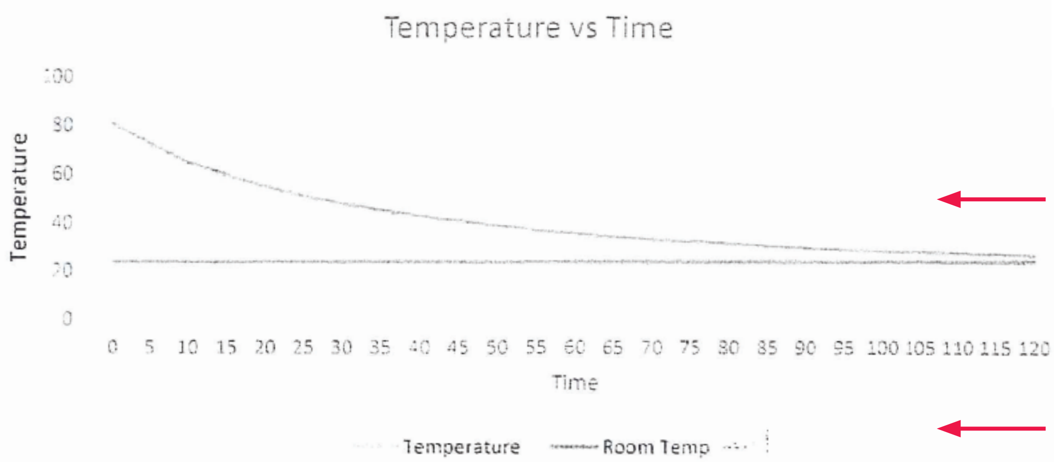
Temperature of Tea													
Time (mins)	0	5	10	15	20	25	30	35	40	45	50	55	60
Temp (°C)	80.4	72.1	64.7	59.2	54.7	50.8	47.6	44.9	42.4	40.4	38.4	36.8	35.4
Time		65	70	75	80	85	90	95	100	105	110	115	120
Temp		34	32.9	31.9	31	30.1	29.4	28.7	28.2	27.6	27	26.7	26.2

B +

Initial Graphing

The next stage logically was to graph the results to produce a graph of my own like I'd seen on the software. I used Microsoft Excel to produce a scatter graph with time plotted as X and Temperature plotted as Y



Not a scatter graph

Units in table.

Here I made the addition of adding the asymptote of room temperature so that I could display the exponential nature of better to myself.

From this it could be assumed:

The power of the exponential is negative: The graph has a negative correlation, so whatever the power of the function is, it must always be negative.

When time (t) = 0, Temperature T=80.4: The initial temperature of the tea (the y intercept) is 80.4

The room temperature is 24 degrees: The tea can never get colder than this temperature.

E +

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From these observations I started to build a formula for the curve displayed above. I chose the power *e* so that I could solve the equations later when finding the constants of the equation. Hence, using T (Temperature) and t (Time) I deduced:

Parameters

It was exponential decay:

$$T = e^{-t}$$

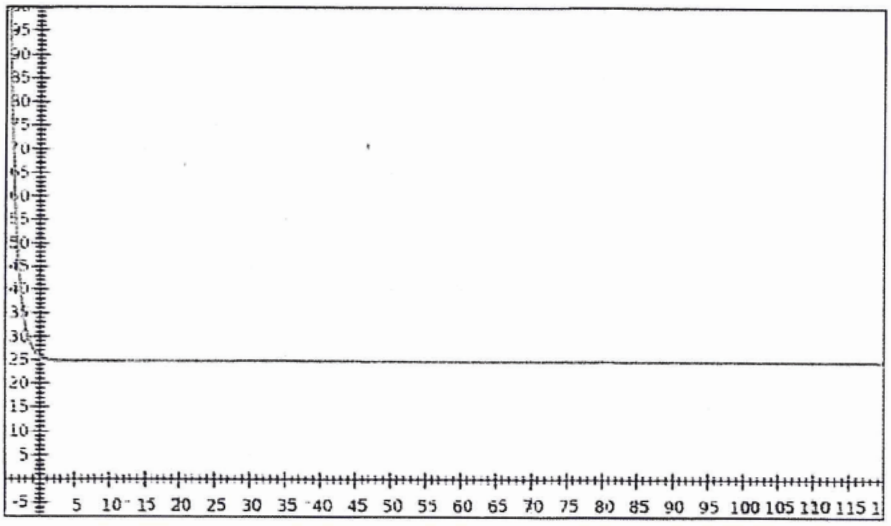
The entire graph was translated up the y axis by a degree of 24, so there was a translation constant:

$$T = Ta + e^{-t}$$

E +

(Where T_a =24, representative of the ambient temperature.)

Drawing this graph produced this:



This is quite blatantly not the correct formula, the axis of the original data were used to demonstrate how where the differences lay between it and the original data. To translate it to represent the original, it requires a translation on the x axis (k) and a gradient much smaller than 1 (a), so it can be assumed that:

-There is a stretch constant on the x axis: a, which is 1 > a > 0

$$T = T_a + e^{-at}$$

B - Two "a's"

-There is a translation constant on the y axis: k

$$T = T_a + ke^{-at}$$

I had lots of issues in this section of the investigation, having produced an equation to solve I found myself at a bit of a loss in how to solve it. My initial line of thinking was that it would be solved using simple algebra and using two results from my data to solve a simultaneous equation.

To start, I rearranged the formula to make e^{-at} more approachable using natural logs:

$$T = T_a + Ke^{-at}$$
$$T - T_a = Ke^{-at}$$
$$\ln(T - Ta) = \ln(Ke^{-at}) \qquad Ln(e) = 1$$
$$\ln(T - Ta) = (\ln K) + (-\mathrm{at}) \times \ln(\mathrm{e})$$
$$\ln(T - Ta) = \ln K - at$$

Then, I let t = 0 and 80 minutes and T=80.4 and 31 degrees

Using two results from the data to create a pair of two variable equations where T_a = 24.5

$$\ln(80.4 - 24.5) = \ln K - 0a$$

$$\ln(31 - 24.5) = \ln K - 80a$$

And solved algebraically:

$$\ln(80.4 - 24.5) = \ln K \qquad \text{A}$$
$$\ln(31 - 24.5) = \ln K - 80a \qquad \text{B}$$

Subtracting equation A from equation B

$$\ln(6.5) - \ln(55.9) = (\ln K - \ln K) - 80\mathrm{a}$$

$$\ln(6.5) - \ln(55.9) = -80\mathrm{a}$$

$$\ln\left(\frac{6.5}{55.9}\right) = -80a$$

$$\frac{-2.15}{-80} = a$$

$$a = -0.0269$$

Please note, all calculated values are left to more than 3.s.f as over simplification might result in the model not matching the results. D

Solving for K

$$\ln(80.4 - 24.5) = \ln K - (0 \times -0.00269)$$

$$\ln(55.9) = \ln K$$

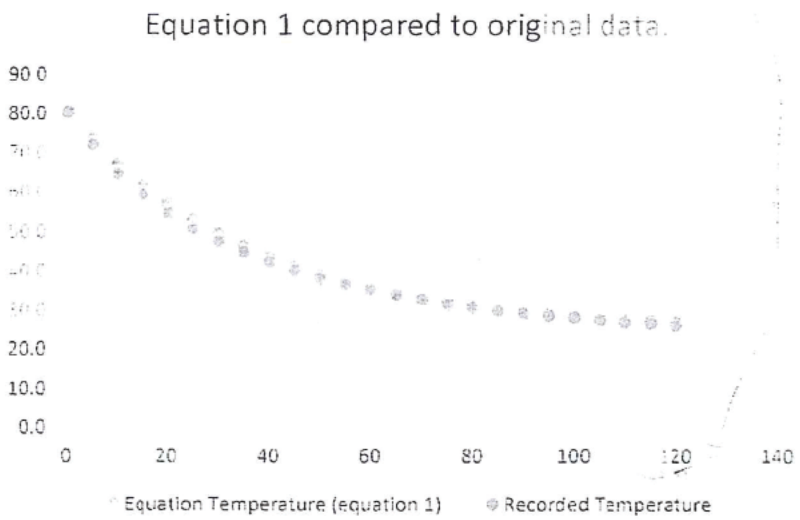
$$55.9 = \mathrm{K}$$

Producing the final equation:

$$T = 24.5 + 55.9e^{-0.0269t}$$

This seems to produce a graph that matches the original data, but it can been seen that the results tend to under predict the rate of cooling in the first 50 seconds of cooling.

$$T = 24.5 + 55.9e^{-0.0269t}$$



It can be seen that the actual data (orange) tends towards being slightly lower than the temperature projected by this first equation. This could have been caused by the method above not accounting for anomalies in the data, as a real world example of cooling isn't going to be perfectly accurate. To try and produce a more accurate series, one that accounts for the natural errors in the data lines of best fit could be used to produce a model of the curve. This attempt follows, and the equation produced above algebraically will be referred to as equation 1

D +

$$T = T_a + ke^{-at}$$

To create a formula that accounted for the uncertainties/anomalies I began to use lines of best fit.

I started to look at the equation in a more representative fashion, the exponential is very difficult to solve in its original form, but straight line graphs are much simpler to solve. With some guidance I once again simplified the equation using logs until it represented something similar to y=mx+c.

Initial equation:

$$T(t) = T_a + ke^{-bt}$$

Typing mistake? (b)

$$T(t) - T_a = ke^{-at}$$

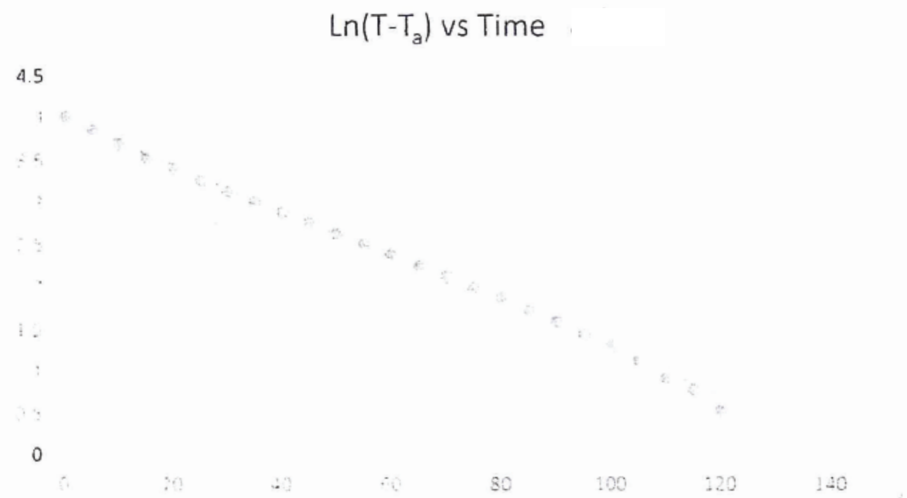
$$Ln\left(T - T_a\right) = Ln(ke^{-at})$$

$$Ln(T - T_a) = Ln(k) - atLn(e)$$

$$\underbrace{Ln(T - T_a)}_{y} = \underbrace{-at}_{mx} + \underbrace{Ln(k)}_{c}$$

$$y = mx + c$$

From this I used expe~~rimental~~ the experiment data to plot the graph of Ln(T-T_a) on excel, producing the following straight line graph.



The line can be seen to be accounting for the small fluxes in the line created by real world factors

Due to the limitations of excel the graph was not produced in enough detail to be able to work out the equation manually, however the program produced a line of best fit and displayed the equation of that line on the graph. It is shown below.

y = -0.0274x + 3.9982

Using this it can be deduced from y=mx+c that:

y = -0.0274x + 3.9982

m =-0.0274

c= 3.9982

Working backwards, these values correspond to the simplified equation

$$\underbrace{Ln(T-T_a)}_{y} = \underbrace{-at}_{mx} + \underbrace{Ln(k)}_{c}$$
$$y = mx + c$$

So that: $Ln(k) =$ 3.9982

$-a =$ -0.0274

$x = t$

From here K can easily be solved:

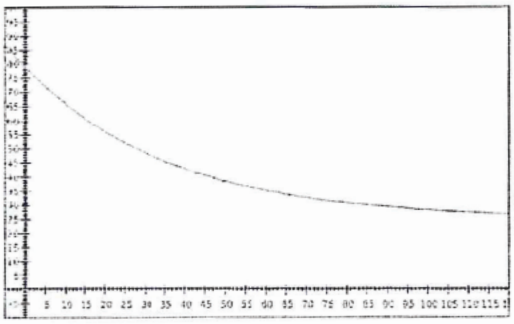
$e^{3.9982} = k$

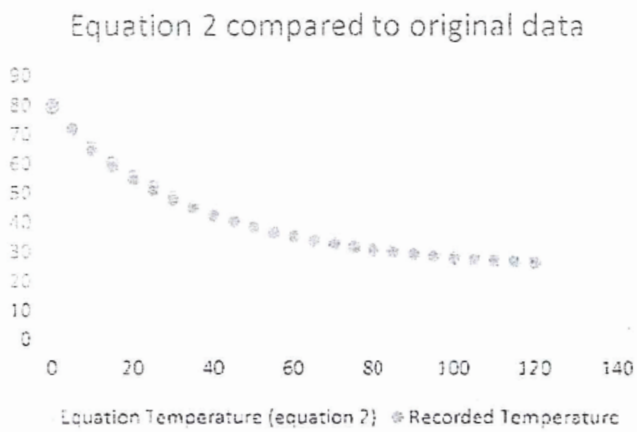
$54.5 = k$

Returning these values to original equation produces

$$T = 24.5 + 54.5e^{-0.0274t}$$

This equation looks more reasonable than the one produced in the algebraic method, graphing it with GraphSketch produces this:





In comparison to the original graph (right) it's virtually identical, crossing the y-axis at what appears to be a slightly lower value. The asymptote is the same, as also appears to be the rate of decay. However to produce a less subjective indication of the accuracy of the model, a full list of values that are produced by the equation, are given overleaf and compared to the real values of the original experiment. A mean average of the differences between the original value for temperature and the one produced by the equation are also attached to provide an easier comparison **D +**

From that it can be quite firmly seen then, that the correlation between the two is very similar, the margin of absolute error between results averaging at a mere 0.11°C. However in discussion with my teacher there is an alternate route to work out the equation of the formula that I would like to pursue to see whether a more theoretical route produces a more accurate equation. Namely the equation produced by Newton's law of cooling. Due to the line of best fit used in this equation it remains to be seen whether or not that has increased or decreased the integrity of the formula by taking into account the discrepancies from the formula produced in the real life experiment, though it appears that it increased the general accuracy of each point, but not the y intercept **C +**

From here on the equation produced in this section by using Lines of best fit will be referred to as equation 2

Comparison of Model and recorded data Equation 2			
t	Recorded Temperature	Equation Temperature (EQ 2)	Difference between values
0	80.4	79.0	1.40
5	72.1	72.0	0.08
10	64.7	65.9	-1.24
15	59.2	60.6	-1.43
20	54.7	56.0	-1.31
25	50.8	52.0	-1.17
30	47.6	48.5	-0.86
35	44.9	45.4	-0.49
40	42.4	42.7	-0.31
45	40.4	40.4	0.02
50	38.4	38.3	0.05
55	36.8	36.6	0.22
60	35.4	35.0	0.37
65	34.0	33.7	0.32
70	32.9	32.5	0.39
75	31.9	31.5	0.42
80	31.0	30.6	0.41
85	30.1	29.8	0.29
90	29.4	29.1	0.27
95	28.7	28.5	0.16
100	28.2	28.0	0.18
105	27.6	27.6	0.03
110	27.0	27.2	-0.18
115	26.7	26.8	-0.13
120	26.2	26.5	-0.33

Mean error:	0.11˚C

$$T = T_a + ke^{-at}$$

In my research to try and solve the cooling equation I discovered Newton's Law of Cooling, which states that:

'The rate of change of the temperature $\frac{dT}{dt}$, is (by Newton's Law of Cooling) proportional to the difference between the temperature of the soup $T(t)$ and the ambient temperature T_a' [1]

$$\frac{dT}{dt} \text{ is proportional to } (T - T_a)$$

The website referenced details the progression from this fact to produce an equation. It uses integration and the fact that T_a and T_o (original temperature) are constants to convert the above integral into a y=mx+c equation, however for the sake of conciseness I will state only the resulting equation that they have produced.

$$T(t) = T_a + (T_o - T_a)e^{-kt}$$

Where: T = Temperature

T_a = Ambient temperature

T_o = Initial temperature

t = Time

k = constant

B +

Factoring in the conditions from my investigation the equation looked like this:

$$T(t) = 24.5 + (80.4 - 24.5)e^{-kt}$$

So, once more using experimental data I let T = 40.4 and t = 45,

$$40.4 = 24.5 + (80.4 - 24.5)e^{-k\times 45}$$

And solved for k

$$\frac{40.4 - 24.5}{80.4 - 24.5} = e^{-k\times 45}$$

$$Ln\left(\frac{40.4 - 24.5}{80.4 - 24.5}\right) = -k \times 45$$

$$\frac{Ln\left(\frac{40.4 - 24.5}{80.4 - 24.5}\right)}{-45} = k$$

$$k = 0.0279387838$$

Producing the equation:

$$T(t) = 24.5 + 55.9e^{-0.0279t}$$

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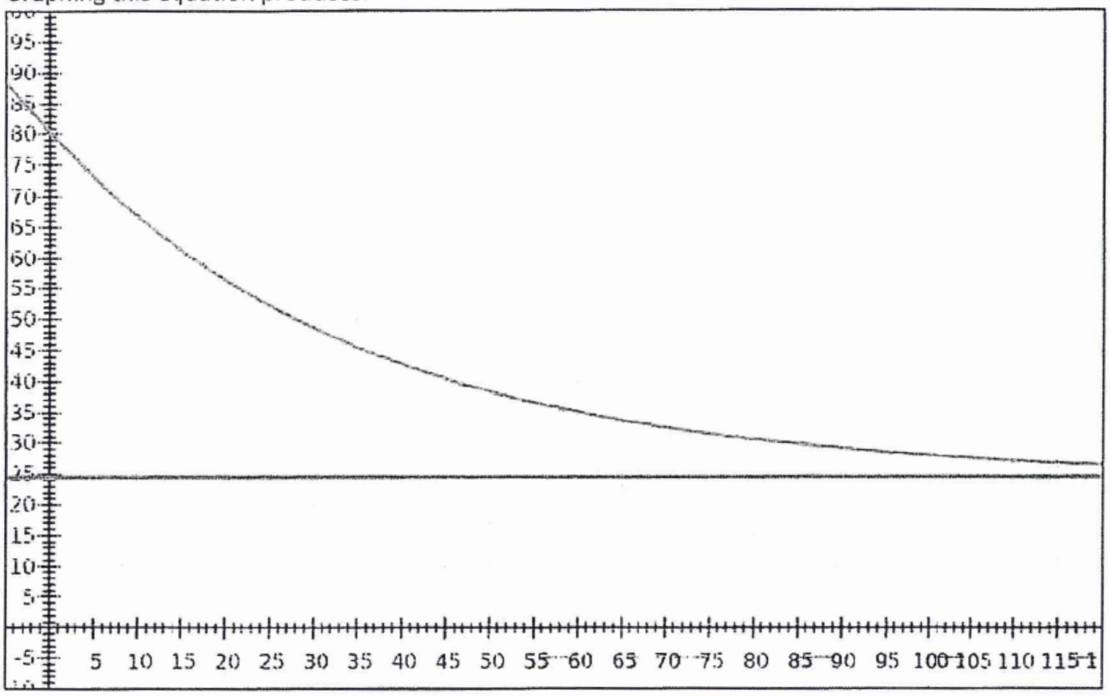
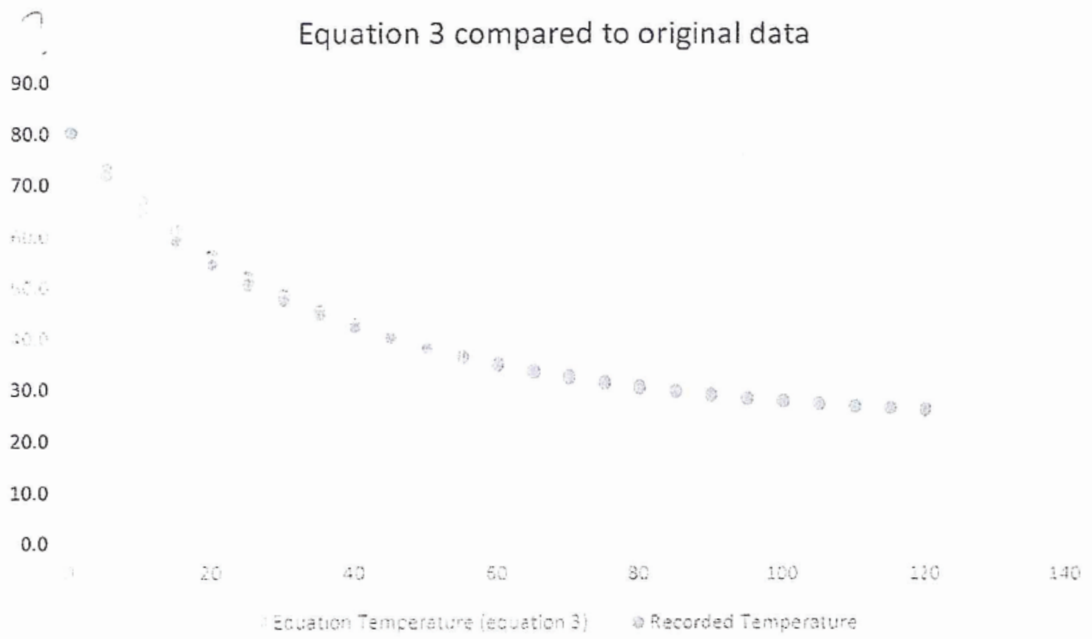
Graphing this equation produces:



Which, when compared to the original graph:



Once again the graphs are almost identical, and the equation produced via this method carries strong resemblance to equation 1 in underestimate of cooling. To determine which gives a more consistent representation of my tea cooling down, a full list of values for equations 1, 2, and 3 are listed, and compared to those recorded in the original experiment.

D +

Comparison of results				
Time	Equation Temperature (equation 1)	Equation Temperature (equation 2)	Equation Temperature (equation 3)	Recorded Temperature
0	80.4	79	80.4	80.4
5	73.4	72	73.1	72.1
10	67.2	65.9	66.8	64.7
15	61.8	60.6	61.3	59.2
20	57.1	56	56.5	54.7
25	53.0	52	52.3	50.8
30	49.4	48.5	48.7	47.6
35	46.3	45.4	45.5	44.9
40	43.6	42.7	42.8	42.4
45	41.2	40.4	40.4	40.4
50	39.1	38.3	38.3	38.4
55	37.2	36.6	36.5	36.8
60	35.6	35	35.0	35.4
65	34.2	33.7	33.6	34
70	33.0	32.5	32.4	32.9
75	31.9	31.5	31.4	31.9
80	31.0	30.6	30.5	31
85	30.2	29.8	29.7	30.1
90	29.5	29.1	29.0	29.4
95	28.8	28.5	28.4	28.7
100	28.3	28	27.9	28.2
105	27.8	27.6	27.5	27.6
110	27.4	27.2	27.1	27
115	27.0	26.8	26.7	26.7
120	26.7	26.5	26.5	26.2

B +

Comparison

Average difference (equation 1)	Average difference (equation 2)	Average difference (equation 3)
0.79	0.11	0.27

The average error (here given as absolute error) shows that equation 2 is more frequently closer to the original value produced by the graph. The least accurate being equation 1, despite it being produced by a virtually identical method to equation 3. This is thought to be due to the values selected which were used to solve the equation. As the real data produced showed some discrepancies to Newton's law of cooling due to real world factors (e.g room temperature changing) , using one of the more anomalous data points to solve and equation algebraically would create a model that not a true representation of the cooling curve.

D +

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It can been seen from these averages that equation 2, the equation produced using lines of best fit from excel have created the most accurate model of the tea cooling down. This is most likely due to the lines of best fit taking into account the slight discrepancies in the real results that were caused by the environment. For example, the third equation relies on the fact that T_a is always equal to 24.5, however as the tea cooled down the air immediately around it would've warmed up as heat diffused away from the cup. This would've created a smaller difference between the temperature of the tea and the ambient temperature, so the rate of cooling would be smaller. This explains why the average difference for equation 3 is so great, Equation 3 always predicts a lower ambient temperature and thus wrongly assumes the rate of cooling is faster, producing a temperature that is (on average) smaller than what the temperature actually was. **D +**

Equation 1 produces such a large error due to, mainly the same reasons as above, however because the algebra used to solve it relied on two randomly chosen values from the real data, any anomalous data from those two values would've been factored into the algebra, so that the error is carried much further forward than in the other two methods of solving the equation.

To answer my original question then: 'How long can I revise before my tea is undrinkable?'

Using my equation:

$$T = 24.5 + 54.5e^{-0.0274t}$$

And assuming that 30°C is undrinkable tea temperature:

$$30 = 24.5 + 54.5e^{-0.0274tt}$$

$$\frac{30 - 24.5}{54.5} = e^{-0.0274tt}$$

$$ln\left(\frac{30 - 24.5}{54.5}\right) = -0.0274t$$

A + *Returns to aim.*

$$\frac{ln\left(\frac{30 - 24.5}{54.5}\right)}{-0.0274} = t$$

$t = 83.7$ minutes

I can safely revise for 1.4 hours before my tea is cold and undrinkable.

As precise as this value is however, it only represents how long I can leave my tea in a 24.5°C room before it is cold. The limitations of the equation include the fact that it doesn't take into account the insulation of the cooling body (namely which mug the tea is in) nor does it account for the fact that an ambient temperature can move. For further investigation it would be interesting to see if an equation can be produced that takes into account the surface area of the cooling body and the changing ambient temperature. Perhaps by plotting the time taken to reach a specified temperature against surface area of cooling body a graph could be produced that could be solved as the one above has been to give a clearing indication of cooling time. However to do so would be to investigate bi-variant data which makes comparisons of the equations more problematic.

D +

However, the equation produced above, despite its minor discrepancies has produced an extremely accurate model of my cup of tea cooling down. I have fulfilled my goal to take the more abstract areas of maths, like the laws of logarithms and solving simultaneous equations which beforehand seemed inapplicable to everyday life and used them to create a value that is both real and relevant to my life.

References:

1. UBC Grad Math. (Unknown). *Newton's Law of Cooling.* Available: http://www.ugrad.math.ubc.ca/coursedoc/math100/notes/diffeqs/cool.html. Last accessed 27th December 2013.

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1. Geoffry Neuss (2010). *IB Course Companion -Chemistry*. 2nd ed. Singapore: Oxford University Press.
2. Buchanan et al (2012). *Mathematics Standard Level*. Singapore: Oxford University Press.