open different perspectives and, we hope, invite exploration. This book, in a few pages, can only gesture toward the territory and hand you (metaphorically) a map. It is up to you, if you will, to do (metaphorically) the exploring.

Mathematics

Have the lights brightened? Have the shadows melted away? Far away, as perfect circles spin in crystal spheres, celestial music sounds in perfect harmony. On a screen the shifting shapes of symmetries and sequences, intricate filigrees, white out softly and vanish into pure abstraction. We have entered the realm of mathematics where the rational mind is at work, at play.

These images may not be what come first to your mind as you do your math assignments. Is there any other picture or image that conveys your experience in this area of knowledge?

> **Activity**
>
> Choose three words that for you best describe the essence of mathematical knowledge. Share them with classmates. Do you find that within your group words and ideas recur? Do you think that the group impression of mathematics is a sound general picture—or a stereotype? How would you go about finding out?

When most abstracted from the world, mathematics stands apart from other areas of knowledge, concerned only with its own internal workings. It retreats, it seems, to the most remote of the ivory towers, in order to think in peace, undistracted by the world.

But for all its removal into abstraction, mathematics, at other times, also gets around companionably in the world. It has developed intimate relationships with other areas of knowledge, helping them to think, express ideas, draw new connections, model the real world, and create new knowledge. It becomes almost part of the family in the natural sciences and the human sciences and is welcomed in professions as various as engineering, veterinary medicine, marketing, and architecture.

You have probably welcomed it yourself into your own family home. From managing a budget to managing your time; from filing income taxes to deciding how much to trust an experimental medical treatment; from calculating the amount of carpet you'll need to purchase for your living room to estimating the ingredients for a shopping list, you may already have found reason to appreciate mathematics. Even if you go into a field that relies minimally on mathematics, being an educated adult in modern society will ensure that mathematics will permeate many aspects of your personal life.

Mathematics gives a splendid entry point into our TOK areas of knowledge.

Is mathematics "the language of the universe"?

Generally speaking, mathematics is the study of patterns and relationships between numbers and shapes. Symbolic and abstract, it takes us into our minds and back out to the world.

No matter what one believes about the origin of mathematics—some philosophers of math continue to engage in the age-old argument about whether we invent mathematics or discover it as we do scientific laws—it is undeniable that mathematical equations can describe the physical universe extremely well. We can truly feel a sense of wonder that the area of knowledge which takes us closest to abstract thought simultaneously provides us, very often, with the symbolic system with which we can talk most precisely about the world.

Two very special irrational numbers illustrate this amazing connection between the abstract and the concrete. Pi (π = 3.14159...) and Euler's constant (e = 2.71828...) show up in many equations in the natural and human sciences, and within mathematics itself.

Pi appears when we consider the circular shape, and is defined as :he ratio of a circle's circumference to its diameter. It naturally ıppears whenever knowledge about circles and spheres is invoked, :ven within physics formulae.

[he formula $e = 1/0! + 1/1! + 1/2! + 1/3! + 1/4! + \ldots$ (infinite series) ›rovides one way to calculate Euler's constant, and uses factorials e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1$; $0! = 1$ by definition). As students of :alculus learn, the function e^x has very peculiar properties. It is even nore peculiar that a number calculated with an infinite series would ıaturally appear in equations describing phenomena as diverse as adioactive decay, the spread of epidemics, compound interest, and opulation growth. Finally, within mathematics itself many consider uler's equation, $e^{i\pi} + 1 = 0$, to be one of the greatest equations of ll time. Not only does it uncannily connect the five most important umbers of mathematics (e, π, 1, 0 and the imaginary number i), ut "what could be more mystical than an imaginary number ıteracting with real numbers to produce nothing?"[1]

/e do not know why natural phenomena are so well described by ıathematics, which is sometimes called "the language of the niverse". Novels like *Contact* by Carl Sagan (in the book this is ıade more explicit than in the namesake movie) presume that any .telligent extraterrestrials we encounter will be able to understand ır mathematics. This belief was also shared by the very real ientists who included in the cargo of the Voyager 1 and 2 ·acecraft (launched in 1977 and now moving beyond the solar stem) phonograph disks which require that our mathematics be :ciphered.[2]

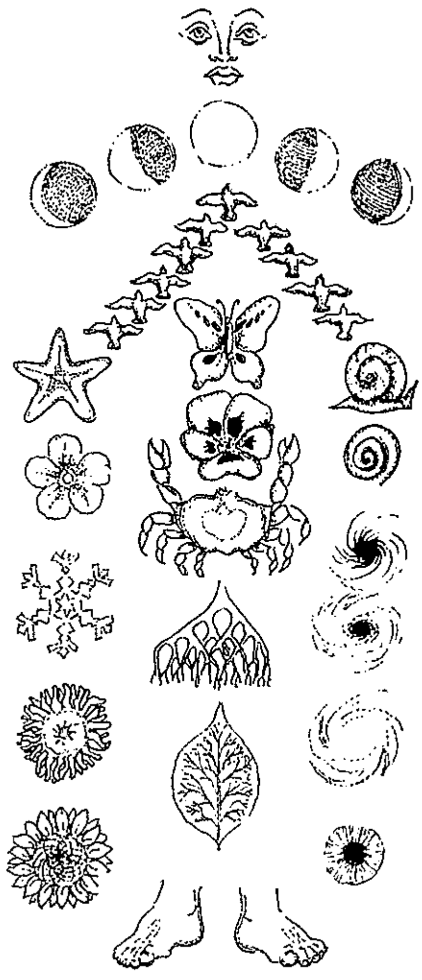
.ıre and applied mathematics

ıe main difference between pure and applied mathematics, as me universities classify their departments, is in the application of ε knowledge they develop. (The qualifier "pure" to describe one ıd doesn't imply that the other kind is impure or inferior; :ording to one practitioner,[3] a more fitting name might be ıeoretical mathematics".) Researchers in pure mathematics— ıich includes abstract fields such as algebra, analysis, geometry, mber theory, and topology—are not concerned with the direct ıctical applications of their labour. Applied mathematicians, on the ıer hand, focus on developing mathematical tools to enable and ıance research in other areas of knowledge. Applied math fields lude numerical analysis, scientific computing, mathematical ysics, information theory, control theory, actuarial science, and ny others.

As is usual with classification schemes, some of the distinctions between "pure" and "applied" are fuzzy. The very establishment of applied mathematics resulted from the successful application of pure mathematics to real-world problems. As Nikolai Lobachevsky once said, "There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world."[4] In the 1970s his assertion was verified yet again with the application of the fundamental theorem of arithmetic—considered useless for more than 2,000 years!—to cryptography, in order to enable secure electronic communications.[5]

A second degree of fuzziness occurs when we consider the distinction between applied mathematics and the areas of knowledge they support. For example, many advances in physics—perhaps even most advances—did not result from fitting a mathematical expression to experimental data points. To derive the equation $E = mc^2$, for example, Albert Einstein applied Lorenz transformations to what he believed was true about light and logically deduced, one step following the other, his theory of special relativity. Thus, it is sometimes difficult to distinguish clearly between applied mathematics and theoretical physics. With the pervasiveness of computational techniques applied to modelling and simulation in various fields, today the boundaries have become even more blurred.

Whether we're speaking of pure or applied mathematics, both deal solely with ideas, at a level of extreme abstraction. The number 2 symbolizes not just two objects of any sort but the idea of two-ness, and the place of two-ness in a number line of other abstractions going to infinity and back to negative infinity—an idea even more abstract. In set theory there can be an infinite number of infinities, and mathematicians can manipulate them through the symbols and the rules they've established to govern their use.



A mathematical world?

The world abounds with patterns that can be described in mathematical terms. How would you describe the examples pictured here?

top: face, the phases of the moon, the pattern of a flock of birds in flight
left column: starfish, buttercup, snowflake, sea anemone, sunflower
middle column: butterfly, pansy, crab, drainage pattern (or...?), leaf, feet
right column: snail, cochlea of the inner ear, water going down a drain, hurricane viewed from above, galaxy, iris of the eye.

To what extent is the naming system in biology affected by characteristics of living things describable in mathematical terms?

In these examples, what kinds of patterns appear within all three categories of animals, plants, and non-living things?

What other mathematically describable patterns in nature can you add to these examples? What, for example, is the Fibonacci series?

The sketch on p. 136 starts and ends with parts of the human body, mathematically describable in number and symmetry. Do we interpret the world as mathematical because we ourselves can be described so, and hence are inclined to see the world in our own terms? Or are patterns describable in mathematical terms part of the world independent of our own minds—a mathematical world of which we are only a part? Even without expecting to answer it, you may find the question quite intriguing, as have others before you.

Mathematics as a language

You have learned many mathematical symbols in your lifetime—all the numbers you can imagine, and many others. Take a few moments to write down ten mathematical symbols (other than real numbers, which would be too easy). Note how each symbol has a very precise meaning.

Are you allowed to combine these symbols in any way you wish? No. In the same way that the string "there go pretty me went I" uses English symbols but is not grammatical, a string such as "x + 2)(= >" is not grammatical. It is meaningless.

Because it is symbolic and can be manipulated into meaningful statements, mathematics has many characteristics of language. Although it does not have the range of functions of language and, arguably, depends on being consciously taught through language, mathematics has features which make it far superior to language as a symbolic system for abstract, rational argument:

1 It is precise and explicit. 3 is always 3, whereas "a few" can mean two, three...or even many, as in "I just need a few minutes!"
2 It is compact. Considerable thought can fit into a few lines. To see the difference for yourself, explain the Pythagorean theorem, $c^2 = a^2 + b^2$, using English.

Now, write down a few of the rules with which you manipulate mathematical symbols and statements. Examples include the commutative property, cross-multiplication, not dividing by zero, reducing a fraction, factorizing a polynomial, and many others. Note that these rules are general. The introduction of rules leads us to two more features of mathematics as a symbolic system:

3 It is completely abstract. It manipulates its statements solely with its own rules.
4 In a way similar to a valid deductive argument, mathematical statements can be manipulated in a step-by-step fashion according to clearly defined rules, leading to new conclusions that were not readily apparent.

Note that when these abstractions are applied to the world, the meaning of mathematical statements gains a concrete dimension. I can abstractly know that the equation $c^2 = a^2 + b^2$ is applicable to every right triangle, but when I'm buying fencing for my garden, determining the shortest drive between two points, or calculating the resultant force in a physics problem, *a*, *b* and *c* have different and very specific meanings.

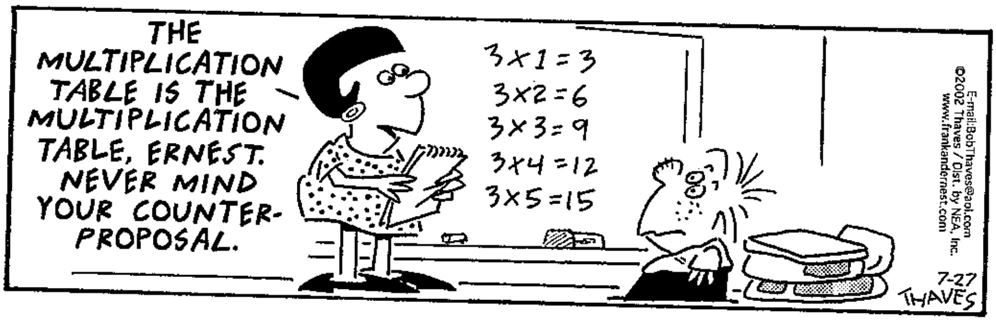
For mathematicians, this precise, compact, abstract, and transformable symbolic system provides the vocabulary and grammar which enable us to talk about abstract relationships such as symmetry, proportion, sequence, frequency, and iteration. Thus, mathematics simultaneously provides a way of analysing not just patterns found in the world by the sciences but also those created from the world by the arts.

1 If you are interested in visualizing mathematical ideas, investigate the artworks of M.C. Escher, which tease and puzzle sense perception while they play with mathematical concepts. You may wish to pair up with someone else in the class or form a small group to look closely at his art and its relationship with mathematics, and share what you have found with the rest of the class. Images and mathematical commentary can be found on the Internet.

2 If you are a musician familiar with compositional analysis, share with the rest of your class some mathematical principles in music. You may wish to pair up with someone else in the class or form a small group to present your ideas. Live performance or recorded music add pleasure to exploration of this topic.

3 Try to write a poem in mathematical language, just for the fun of it. Students before you have done so with results that are quite entertaining. Concepts of nothingness, difference, union of sets, and infinity, for example, seem to lend themselves to poetry—but they are merely the start. When you have exhausted your capacity to unite mathematics with your poetic imagination, consider whether what you have written could be considered mathematics or, rather differently, language using mathematical imagery. Think back also to the comparisons you drew between different symbolic systems in Chapter 2 (see page 33).

With such a wide range of applications in the world, what is it that mathematics cannot do as a "language"? Why is the IB never likely to offer mathematics A as a group 1 subject?



Mathematics is frequently spoken of as having foundations, rather like a building with a strong stone or concrete base. If the foundations are solid and unshakeable, the construction that is built on top rests secure. For mathematics, assumptions known as axioms provide the foundation, and through the process of deductive reasoning, step by step, often over a span of many centuries, mathematicians carefully erect a building.

Different mathematical fields such as geometry, algebra, set theory and number theory are axiomatic, deductive systems. Each of these fields is based on a different set of axioms, but relies on the same method to develop new knowledge. By using the axioms at the foundation as premises and applying valid deductive reasoning to them, mathematicians obtain—through a process called mathematical proof—new statements called theorems. These, in turn, are used as additional premises to build further theorems, which are in turn used as additional premises...ultimately giving rise to entire structures consisting of interconnected mathematical statements.

But what is missing from this picture? Think back to reasoning as a way of knowing and the blender analogy discussed in Chapter 2. Recall that in order to generate true conclusions, a deductive argument requires not only valid reasoning, but also true premises. How do we know if the axioms used by mathematicians as the foundations of their structures are true? That question turns out not to be simple. Actually, it is a critical knowledge issue in mathematics.

Historically, geometry was the first axiomatic, deductive system to be developed. It was Euclid, 2,300 years ago, who identified the first known set of axioms, only ten of them (the fewer, the better!). He considered these axioms (which he called "postulates" and "common notions") to be true, derived from experience and requiring no proof.[6] With one proof at a time—some less formal than others, because Euclid "assumed details and relations read from the figure[s] that were not explicitly stated"—Euclid's system of plane geometry was built. Students worldwide continue to study it in schools today.

For over 2,100 years, Euclidian geometry was considered to be perfect knowledge. It was regarded not just as valid but as true—true not only in its logical consistency (coherence test for truth) but also true in the world (correspondence test for truth). Even more significantly, Euclidean geometry was considered to be eternally true.

No challenge came to the perfection of Euclid's mathematical system until the 19th century, and even then the challenge was not to its validity but to its truth. What if Euclid's axioms, the very foundations of his system, were not true—or were not the only possible truth?

The first four of Euclid's axioms seemed self-evident because they could be verified by drawing figures on the sand. The first required joining two points with one, and only one, line segment; the second required imagining that this line segment continues forever on the flat ground; the third required constructing a circle centred on a point; and the fourth required only that people compare right angles they could easily draw, and conclude that the angles are congruent. But the fifth axiom—known as the "parallel postulate"—was more problematic, even for Euclid, who only invoked it upon proving his 29th theorem.[7] How could anyone ensure that you can draw only one line through a point P that is parallel to a given line? Verifying the truth of that axiom would require someone to accompany the line forever, to ensure that it never intersects the first line. Mathematicians tried to prove the fifth postulate as if it were a theorem, and failed.

Independently of these quirky little technical problems, countless generations benefited from knowing one of the theorems proved by Euclid, that the sum of the angles of a triangle is 180°. They found in geometry's established truths easy ways to solve their everyday problems, such as determining how much wall to build around a perimeter or calculating the area of their fields. Meanwhile, mathematicians continued to struggle with the fifth postulate, mainly by trying to prove that Euclid's system was foolproof.

It was Carl Friedrich Gauss in the early 1800s who first noticed that a geometry could be built without including Euclid's fifth postulate. Gauss paved the way for the non-Euclidean geometries of Nikolai Lobachevsky and later that of Bernhard Riemann. Lobachevsky replaced Euclid's fifth postulate with the idea that through a point P next to a given line, at least two lines exist that are parallel to it. Riemann, on the other hand, assumed that no parallel lines exist through P, which logically implied that he had to adopt modified versions of Euclid's first and second postulates as well. (To understand why, imagine Riemann's geometry happening on the surface of a sphere instead of on an infinite plane surface like Euclid's. On a sphere's surface, more than one line can be drawn between two points, and lines cannot be extended indefinitely.[8]) These non-Euclidean geometries—consistent and valid, though based on different axioms—shook the very foundations of mathematics.

Mathematics: definitions and playing by the rules

> A farmer called an engineer, a physicist, and a mathematician and asked them to fence the largest possible area with the least amount of fence.
>
> The engineer made the fence into a circle, and proclaimed that he had the most efficient design.
>
> The physicist built a long, straight line of fence and proclaimed "If we were to extend this length around the Earth, we would have the largest possible area."
>
> The mathematician just laughed at them. He built a tiny fence around himself and said, "I declare myself to be on the outside."

It had been assumed that Euclidean geometry was true according to the correspondence truth test, accurately describing space. How could other consistent geometries be built using different axioms, geometries that didn't have any bearing on reality? Were mathematical systems not necessarily true? The answer to these questions changed the whole notion of mathematical truth. Mathematical truth came to be understood as truth within a system: mathematical statements could be true within the Euclidean system, or true within the Riemann system. The only truth test relevant was the coherence test, or in other words the consistency of every statement with every other statement within its own axiomatic system.

Though Lobachevsky's geometry hasn't been shown to apply to the cosmos, in 1916 Riemann's geometry did find a practical application. The curved space of Einstein's theory of general relativity is well described by Riemann's geometry.

We now consider axioms to be not "self-evident truths" but to be the assumptions, premises, definitions, or "givens" at the base of a mathematical system. We still use the metaphor of foundations, but recognize more than one possible construction. Euclid's geometry is more useful in building a house, Riemann's is more useful in flying an airplane, and Lobachevsky's, in accordance with his own quoted words a few pages ago, might one day find a practical application…or not.

With the failure of Euclidean geometry to describe physical space as had been expected, a vast amount of room opened up for the creativity of mathematicians. Today, they do indeed have the freedom to declare whatever they please, independently of whether their assumptions have any bearing on the real world or not.

Once a mathematician adopts any specific set of definitions and rules, however, he must play by them—very, very strictly.

Reflections on mathematics

For class discussion

In what ways does the general public–like you and your family–benefit, directly and indirectly, from the products of mathematical research?

Make a list of the ways in which you're classified based on the numbers associated with you (e.g. your number among other telephone owners). What parts of your life do numbers enter? If you seem to have more numbers attached to you than others do, what has created this difference?

With the pervasiveness of computers, might we as a culture have become too attached to representations of the world in quantitative terms? Consider the following statement about world hunger: "…how we understand hunger determines what we think are its solutions. If we think of hunger only as numbers–number of people with too few calories–the solution also appears to us in numbers–numbers of tons of food aid, or numbers of dollars in economic assistance."[9] Do you agree this might be the case? What would support this argument? What would counter it?

For research and class discussion

Identify three formulae or algorithms which you find interesting in your current mathematics textbook. Research who developed them, and when and where they were developed. Why did you choose these specific three? Do they have anything in common? Share your insights with classmates.

Investigate some of the specializations in which applied mathematicians collaborate with researchers in other fields. Does any of these fields appeal to you? Does it surprise you that being a mathematician doesn't necessarily imply working within a university?

Mathematical proof: challenging and beautiful

Euclid and Riemann both created knowledge by means of the characteristic method of justification in mathematics: the proof. To create a proof, as we have seen, the mathematician takes as his premises the foundational axioms and all subsequent theorems and proofs based on them. Then, with a problem or conjecture in mind, he reasons toward a new conclusion, taking immense care to avoid error in any step. In manipulating ideas in a process of pure thinking, he creates new knowledge. That new theorem, in turn, provides a base for further reasoning.

Mathematicians, taking pleasure in such abstract creation, are the more delighted if the proof goes beyond merely being valid. It should be, as they say, elegant. The elegant or beautiful proof is incisive and ingenious. It is economical in using as few steps as possible and holds a little jolt of surprise as ideas fall neatly into place. A swirl of a cape, a flash of a rapier and—voilà—proved! Or so the mathematician would like.

When we liken mathematics to a game with its own internal rules, we do not mean that it is trivial. Games can be very serious.

However, mathematicians are not always very serious. What is the invalid step in this proof?[10]

What's wrong with this proof?

Given: $A = B$

Multiply both sides by A: $A^2 = AB$

Subtract B^2 from both sides: $A^2 - B^2 = AB - B^2$

Factorize both sides: $(A + B)(A - B) = B(A - B)$

Divide both sides by $(A - B)$: $A + B = B$

Since $A = B$, $B + B = B$

Add the Bs: $2B = B$

Divide by B: $2 = 1$

A new proof, no matter how beautiful it is, does not enter the realm of mathematics until it becomes public knowledge: the truth of the claim must be justified to the relevant knowledge community which, through the process of peer review, must come to believe the claim's truth.

A good example of peer review at work is the rejection, for almost four centuries, of all attempted proofs for what came to be known as Fermat's Last Theorem (FLT).

In 1637, Pierre de Fermat, as the story goes, was reading for pleasure a book of ancient mathematics, a French translation of Diophantus' *Arithmetica*. Mathematicians still do not know what was going through his mind when he wrote in the margin of the book the message, "I have a truly marvellous demonstration of this proposition which this margin is too narrow to contain."[11] Without ever sharing his proof, he died. Published posthumously by his son in 1655, the note remained. Fermat had a solid reputation as a mathematician, so it could not be dismissed lightly. But what was his "marvellous demonstration"? Fermat had left to his successors the most famous unsolved problem in the history of mathematics.

We know from working with right triangles that many trios of integers can satisfy the equation $c^2 = a^2 + b^2$. What Fermat postulated was that no trios of integers exist that can satisfy equations such as $c^3 = a^3 + b^3$, or $c^4 = a^4 + b^4$, and so forth, for powers greater than squares. Many mathematicians tried and failed to find a proof. Even more just turned away to work on problems more likely to be fruitful. Why waste time on FLT?

When a proof was announced, it caused a sensation. It was in 1993 that British mathematician Andrew Wiles first announced he had proved FLT. Wiles presented his 150-page paper at a conference as a "traditional mathematical proof", which omits routine logical steps and assumes that knowledgable readers can fill in the gaps. Such proofs rely on intuitive arguments which can be easily translated by trained mathematicians into rigorous deductive chains. Proofs are usually presented this way because too much formality would obscure its main points, much like watching a movie frame by frame would distract the viewer from enjoying, or perhaps even understanding, its storyline.

Peer review went to work—and this version of Wiles' proof was found to have a flaw. In Wiles' own words, "It was an error in a crucial part of the argument, but it was something so subtle that I'd missed it completely until that point. The error is so abstract that it can't really be described in simple terms. Even explaining it to a mathematician would require the mathematician to spend two or three months studying that part of the manuscript in great detail."[12] Wiles went back to work, creating still more mathematics in order to remedy the error.

In 1994 Wiles presented his amended proof. Again peer review went to work—and this time the mathematical community accepted the proof. Wiles became a celebrity overnight, surrounded by public excitement over the solution of such a famous and longstanding problem. Intriguingly, though, his proof of Fermat's Last Theorem cannot have been Fermat's own, as the 20th-century mathematics on which it is based was unknown, back in 1637, to Fermat.

The story of this proof illustrates many characteristics of mathematics as an area of knowledge. For one thing, it shows something of its humanity—the fascination, the challenge, the creativity, the aspiration, the disappointments, the sense of triumph. At the same time, though, it reflects characteristics of more ordinary mathematical endeavour—the level of care and detail demanded, the peer review and its difficulties when the work is new and complex, and the respect given to achievement that the lay public does not understand and for which there may be no apparent practical use.

In 2006, Wiles' proof of FLT has not yet been developed into a rigorous or formal proof, showing every single deductive step. Computer scientists have now been challenged to "formalize and verify" it, and one of them estimates that he expects this problem to be solved in about 50 years.[13]

(Indeed, contemporary mathematical proofs are rarely brief. In 2003, Russian mathematician Grigory Perelman announced that he had solved a classical problem within the field of topology, the Poincaré conjecture. In 2006, Perelman's traditional proof was confirmed after peer review.[14] It was roughly a thousand pages long.)

Clearly, the relationship between the mathematician Diophantus of ancient Greece, Pierre de Fermat of 17th-century France, and

Dr Andrew Spray,
IB Diploma Programme mathematics teacher and examiner

Why do you love mathematics?

The challenge in replying is that it is hard to give one single reason. Among the features that are very endearing are:

- The elegance of the logic behind many classic mathematical proofs.
- The unexpected connections between apparently distinct areas of mathematics. e.g. e appearing in probability problems, π appearing in summing the inverse powers of numbers, the classic $e^{i\pi} + 1 = 0$.
- The beautiful patterns, usually unexpected, that arise between numbers. e.g. 3 cubed + 4 cubed + 5 cubed = 6 cubed (or $3^3 + 4^3 + 5^3 = 6^3$).
- The fact that areas of mathematics developed solely for the sake of pure mathematics turn out to have very useful applications in the real world, for example complex (or imaginary) numbers, group theory.
- The joy of being able to solve a complex problem and prove you are correct.

I love mathematics for all these reasons, and more, but perhaps it is the patterns, the searching for them and the joy of discovering them, that captivate me most.

contemporary Andrew Wiles of Britain (who developed his work in the United States) highlights certain features of mathematical knowledge. Its challenges and its products can last over centuries. Yet once it is satisfactorily proved, the proof is permanent in all places and all time, and the proven knowledge claim earns its place as yet another brick in the edifice of mathematical knowledge, built across boundaries of time and culture.

Placing the spotlight on the successful proofs, however, may obscure the contributions of the failures. Have their failures really been failures for mathematics? After all, the development of mathematics relies on failed attempts at proof as well as successes. Much new knowledge is generated in attempts to solve problems; many interconnections between mathematical fields are established. As Wiles said about his own effort, "The definition of a good mathematical problem is the mathematics it generates rather than the problem itself."[15]

Mathematics and its critics

With the creation of considerable mathematical knowledge through the past century, mathematics is evidently flourishing. However, as an area of knowledge with its characteristic means of justification, mathematics has also faced criticism of its very foundations.

The growth of mathematical knowledge: exercise

by Manjula Salomon

In the following exercise, you will take on a research topic, find out about it, and share your findings. Be prepared to identify your findings according to historical time and place of origin.

Divide your group so that someone is investigating each of the following topics. Allow at least 20 minutes in the library or on the Internet for the investigation. For finding the most crucial details, your best source may be an encyclopedia.

- abacus
- Ramanujan
- calculus
- Omar Khayyam
- geometry
- algebra
- algorithm
- infinity
- decimal system
- probability
- Pythagoras' theorem
- chaos theory
- zero
- Euclid
- trigonometry

Create a timeline on the board or a large poster. Each person or group should report the information obtained and place the relevant information on the shared timeline.

Questions for discussion

1. What interdevelopments do you see between the various topics?
2. To what extent does your research suggest that mathematics is an international area of knowledge? How would you compare it in this regard with other areas of knowledge?
3. Does your research challenge any of your previous assumptions?
4. The development of mathematical knowledge is often illustrated by a tree diagram (that is, roots labelled as arithmetic, the trunk labelled as calculus, etc.). Mathematical scholars often select the banyan tree as the best tree for such an illustration. Why might this be so?

Note: Conventional division of the mathematical history timeline separates it into periods: earlier times to ancient Babylonia and Egypt, the Greek contribution, the Far-Eastern and Semitic contribution, and the European contribution from the Renaissance onward.

Consider now the diagram at the beginning of this chapter for its application to mathematics.

Knowledge Creation Diagram

within context of the natural world

knowledge creator(s)	work(s) of knowledge *(math proof, research paper, novel)*	knowledge community *(peers, critics general public)*

within social context

We have considered the creators of new knowledge, the characteristics of mathematical work, the role of peer review in the process of public justification, and mathematics' relationships with the natural and social world depending on whether we are speaking of pure or applied mathematics.

Let us look more closely now at the role of the critic, who applies critical thinking to a balanced examination of the justifications of the knowledge claims, seeking to appreciate both their strengths and their limitations. The critics of mathematics, those who evaluate each new mathematical work, are themselves mathematicians—peers of the knowledge creator(s). At a higher level of abstraction, though, are those mathematicians who evaluate the entire area of knowledge, examining its knowledge claims for their nature and their bases. Sometimes philosophers, sometimes highly reflective mathematicians, sometimes metamathematicians, these critics concern themselves with such issues as the reliability of the foundations of mathematics and the nature of proof.

In the early 20th century they reached a shocking conclusion: that mathematical knowledge has flaws and limitations, implying that mathematicians do not have an absolutely unshakable basis for their knowledge claims. Mathematicians had thought they had possessed that solid basis before the development of non-Euclidean geometries, and were hoping to restore the status of mathematics as a field providing absolute, eternal truths.

So, after mathematicians realized that mathematical truths must be evaluated using the coherence truth test (which implies that the axioms they use as foundations need to be logically consistent), they turned their attention to studying axioms more attentively. Could mathematics reach a state of completeness—a state in which it would be whole, having all its necessary elements or parts? This translated into another two questions. First, can all propositions be proved or disproved from axioms within the system? Second, can the consistency of the axioms be proved (can we be sure they don't contradict each other)? Bertrand Russell, working with Alfred North Whitehead, had been trying to deduce the entire field of mathematics from the principles of logic alone. They started with arithmetic, by attempting to construct the real number system using mathematical sets as a tool.[16]

In 1901, they were disturbed to discover a contradiction regarding those sets which are, or are not, members of themselves. If the set is a set of chocolate bars, for example, the set is *not* a member of itself. However, if the set is a set of all those things that are not chocolate bars, then the set is a member of itself. Russell discovered that he could easily create a contradiction, no matter what objects he was including in the set, by creating a set of all sets that are not members of themselves. Hence a member of the set would have to be (a) a member of itself, because it is part of the set and (b) not a member of itself, because that is what the set is—a set of things *not* members of themselves.

Russell's paradox had implications for all mathematics: if mathematics is an intellectual game played by its own internal rules, and expected to be complete and free of contradiction, then what claim to knowledge can it have if there is an inconsistency within it? Russell and others, including Gottlob Frege and David Hilbert in the 1920s, attempted without success to eliminate paradox from mathematics.

Verbal analogies to self-reference and contradiction may give some sense of what these mathematicians experienced. Self-reference, after all, is not unusual in itself. Singers sing songs about singing songs, poets write poems about writing poetry, and painters have been known to paint paintings of painters painting. Every time you use "I" you are using self-reference. Even reflection on knowing in TOK is often self-referential. Still more so is the research of cognitive psychologists, who use their brains to think about the thinking of the brain. (If you wore a self-referential T-shirt, what would be the

Jest for fun

Mathematics is made of 35 percent formulae, 35 percent proofs, and 35 percent imagination.

There are three kinds of people in the world; those who can count and those who can't.

design on it? If you took a self-referential photograph, what would it show?)

When self-reference creates contradiction, the results can be quite witty. The writer Oscar Wilde once quipped, "I can resist everything—except temptation" and on the basis of similar cleverness became a favourite party guest for a while. Depending on your sense of humour, you may find paradox quite entertaining as it jams your mind with contradiction: "Disobey this command." (Just try doing that!) Ancient paradoxes live on to perplex us largely because we enjoy them: Epimenides, from ancient Crete, uttered the claim, "All Cretans are liars" or, in another version, "I am lying." Well, if he is telling the truth, does that mean he is lying? If he is lying, does that mean that he is telling the truth? This kind of paradox, many find, is immensely entertaining. But mathematicians did not burst into laughter when Gödel made a similar move in mathematics.

In 1931, Kurt Gödel published what is now known as "Gödel's Incompleteness Theorem", which basically states that the dream of having mathematics reach a state of completeness is impossible to achieve. There cannot be a guarantee, within any axiomatic system, that the axioms adopted will not give rise to contradictions. There will always be, in any formal system, statements that are not decidable within it. Thus, no axiomatic system can ever prove its own consistency.

Gödel had no intention of knocking the supports out from under mathematics—and also its hope of being the only area of knowledge able to achieve absolute certainty because of its reliance solely on reasoning. Gödel intended exactly the opposite, actually—to ground the axiomatic approach to mathematics the more firmly on logic. With considerable ingenuity, though, he followed where his reasoning led him, creating through a numbering system a means of self-reference within mathematics that led to internal paradox and, ultimately, to the Incompleteness Theorem.

Despite having been shocked into the realization that mathematical knowledge has limits, mathematicians survived, and kept on working. The dream of absolute certainty is not attainable in mathematics, nor is it attainable in the natural sciences, as we shall see in the next section. But that doesn't prevent us from learning as much as we can, including learning to judge how much we can trust the knowledge we glean. The revelation of the flaws in mathematics has not stopped mathematics. On the contrary, it has given it a new understanding of itself, new problems to solve, and new directions for the mind.

"A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematicians can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies."

Stefan Banach[17]

G.J. Chaitin, a contemporary mathematician who stated that any given number cannot be proved to be random,[18] recently looked back on the Incompleteness Theorem as almost inevitable—as a step in mathematical progress now absorbed into further thinking. Like Alan Turing's later work and Wiles's more recent proof, for him the Incompleteness Theorem becomes clear in hindsight: "So you see, the way that mathematics progresses is you trivialize everything! The way it progresses is that you take a result that originally required an immense effort, and you reduce it to a trivial corollary of a more general theory!" He speculated that in a century or two, Wiles's proof, hundreds of pages long, will be reduced to a single page and understood readily in the context of mathematics developed after its time. "But of course that's the way it works. That's how we progress."[19] So maybe, after all that is said and done, we'll finally figure out what proof to his last theorem Fermat had in mind when he wrote that note in the margin.

Mathematical progress—perfect or imperfect—surely takes us to its ivory tower, remote from the world in a realm of pure thought. It is a particularly intricate tower, carved and incised with immense care for detail, and elegant in its shape. Within it, pure mathematicians build their proofs with little concern for practicality, while many practitioners of other areas of knowledge wait, hoping that they will produce the mathematical knowledge and language that will be useful within their own fields. The remote tower, after all, has never lost its connection with all the others.

Let us give the final word on mathematics to someone passionate about it, IB graduate (1999) Gergana Bounova of Bulgaria, who concluded an essay in TOK with a personal declaration about this area of knowledge:

> *Ultimately, I am certain about one thing—mathematics is extremely beautiful. Only a few can truly appreciate it. Beauty is not in the eye of the beholder. Beauty is in the mind of the beholder. Mathematics is a sophisticated toy you can play around with until reaching total intellectual satiation. It is unbelievably perfect and this is why I feel it is not the universal language. The world is an interesting but imperfect place and needs something to balance it. So let's dream in mathematics and wake up in the real world.*

...es

The Magic Gopher: www.learnenglish.org.uk/games/magic-gopher-central.swf.

Has the study of this section changed your understanding of, or your feelings about, mathematics and mathematicians? If so, in what ways?

As we move from mathematics to the sciences and history, the clear light becomes increasingly dappled with recognizable shapes of the world—trees, animals, and passing human beings. We have left the realm of pure thought and are entering the world and the areas that study it.

In your diagram or lists at the beginning of this chapter, what connections did you make between physics, chemistry, biology, psychology, economics, anthropology, and history? In TOK, the first three are natural sciences, the second three are human sciences, and history stands on its own. However, for the moment, we will consider all of them together.