4 Mathematics

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
Albert Einstein

MATHEMATICS IS THE SUBJECT WHERE WE NEVER KNOW WHAT WE ARE TALKING ABOUT, NOR WHETHER WHAT WE ARE SAYING IS TRUE.
Bertrand Russell

WHEN YOU HAVE SATISFIED YOURSELF THAT THE THEOREM IS TRUE, YOU START PROVING IT.
Arthur Koestler

. . . it is certain that the real function of art is to increase our self-consciousness; to make us more aware of what we are, and therefore of what the Universe in which we live really is. And since mathematics, in its own way, also performs this function, it is not only aesthetically charming but profoundly significant. It is an art, and a great art.
John W. N. Sullivan

If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be appied in mathematics just the same as in physics.
Kurt Godel

There is nothing that can be said by mathematical symbols and relations which cannot also be said by words. The converse, however, is false. Much that can be and is said by words cannot successfully be put into equations, because it is nonsense.
C. Truesdell

Mathematicians may have trouble talking to non-specialists about what they do – but they also have trouble talking to each other. The idea that maths is some kind of universal language is a myth – mathematicians from different areas simply cannot understand one another.
Anon

Mathematics is created in the self-alienation of the human spirit. The spirit cannot discover itself in mathematics; the human spirit lives in human institutions.
Giovanni Vico

You can not apply mathematics as long as words still becloud reality.
Hermann Weyl

On each decision, the mathematical analysis only got me to the point where my intuition had to take over.
Robert Jensen

Nobody untrained in geometry may enter my house.
Plato

Aims

By the end of this chapter you should:

- **understand the axiom-theorem structure of mathematics**
- **understand the implications of this structure for mathematical truth**
- **understand the role of logic in mathematics and the link to rationalism**
- **be able to discuss possible links between mathematics, science, art and language**
- **understand why mathematics may be regarded as an extremely creative discipline**
- **have some insight into the process of attempting to establish a theorem to describe a situation**
- **understand that the initial promise of the axiomatic approach has been undermined by Gödel, and be able to mention possible implications of his ideas.**

Introduction

It may not be obvious immediately why a book with a philosophical leaning contains a chapter on mathematics. What could be less ambiguous, more clearly defined and less open to interpretation than a mathematical problem? A maths problem may not be resolved easily, but there is a right answer, and little room for debate – we are probably all too familiar with the rather tedious and long-winded maths exercises which are marked right or wrong. So why would we include such a dry topic in a course such as this?

The answer is two-fold. Firstly, the relative certainty of mathematics is exactly the reason we need to include it – if it presents us with indubitable knowledge then we need to learn precisely how it does that and see if we can apply the technique elsewhere. The techniques of mathematics may provide us with a tool that will be central to our search for reliable knowledge. Secondly, we will argue that the stereotypical image presented above is just that – a stereotype. There is far more to mathematics than the rigid application of formal rules to meaningless systems of symbols (although this may, arguably, be the end result). It is creative, imaginative, deeply satisfying and in some ways similar to those disciplines sometimes considered diametrically opposed to mathematics – the arts.

Mathematics is a subject which everyone finds difficult at some stage. There are often negative attitudes to the subject, and these arguably stem from the requirement to learn a large body of knowledge that seems to have few relevant applications. However, maths is also an immensely powerful tool in its application to science (at least) as witnessed by cars, telephones, computers, moon landings, aeroplanes and atomic bombs. It plays a central role in any technology, and is increasingly finding its way into apparently unrelated fields such as history, medicine, psychology, art and music. To some, maths is pointless and irrelevant and they will not bother with it if they can avoid it.

To others it is fascinating and a source of never-ending delight. That a topic can appear in such diverse contexts, and in such different ways to different people – this alone makes it well worth studying.

Mathematics: invention or discovery?

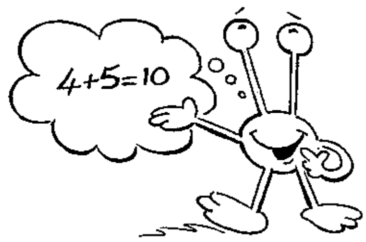
This makes sense. This over here this does not make sense. That's why it's called algebra

Anon

When you solve a mathematical problem, you probably feel like you are **finding** the solution. You may feel that mathematical truths are always true. 2 + 2 = 4; no argument there. And 242,324 is an even number, whether we like it or not. We would be foolish to look for a triangle with seven sides. Consider 345 × 53. You probably can't do it in your head, but you could work it out given a pencil and paper and a little time. Certainly with a calculator the answer could be found quickly. The correct answer doesn't depend on who does it, when they do it or how they do it. They may get it wrong, of course, but the answer itself is always 18,285. We have no choice as to what the answer is. We have to find it.

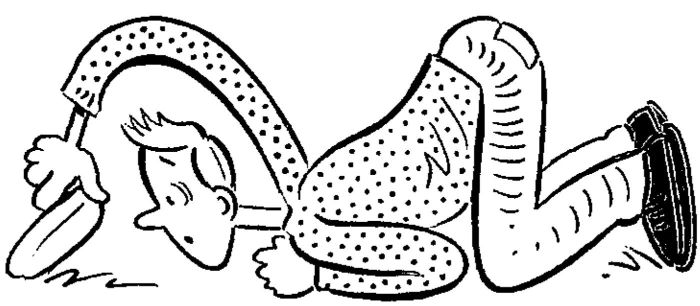
Imagine we contact alien life forms and try to communicate. Will we find that they believe in different mathematical results? Will they have calculated different mathematical answers to us? If they have calculated π, for example, will they have found the same value as us? Will they believe that 2 + 2 = 4?

The view that 'maths is out there waiting to be discovered' is called the Platonic view of maths, after Plato, who thought that mathematical truths are eternal and unchanging. At first sight, this seems very appealing, as we have seen from the examples just given. However, there are some difficult questions for Plato to answer:



- Where does mathematics exist?
- How do we 'discover' maths?
- Why does the 'real world' obey mathematical laws?

These are quite profound problems, because many find that the only reasonable answers tend to suggest that, contrary to what we initially suggested, mathematics is purely in the mind. Now Plato would not have minded this (he argued that we are just 'remembering' things we already knew but had forgotten) but this sort of answer doesn't carry much weight today. If we find that mathematics is really in the mind then isn't it an invention? This may answer the problems mentioned above (how?), but it raises its own difficulties.



Looking for $\sqrt{2}$

- Surely we can't have invented the fact that 2 + 2 = 4? That goes against all common sense! If maths is invented, why don't different mathematicians invent different mathematics?
- If maths is invented, in the same way that artists invent art, how can answers to mathematical questions be right or wrong?

It has been suggested that mathematicians would *like* maths to be discovered – that is how they feel emotionally towards their work. They talk about 'discovering' theorems and this attitude pervades their working life from Monday to Friday. However, if pressed hard on the matter, when philosophising at the weekend, most will retreat away from 'discovery' to 'invention' as they cannot logically justify 'discovery' to their satisfaction. One mathematician who refused to retreat in this manner was G. H. Hardy, one of the great number theorists of the twentieth century. In *A Mathematician's Apology* he wrote:

I began by saying that there is probably less difference between the positions of a mathematician and of a physicist than is generally supposed, and that the most important seems to me to be this, that the mathematician is in much more direct contact with reality. This may seem a paradox, since it is the physicist who deals with the subject matter usually described as 'real'; but a little reflection is enough to show that the physicist's reality, whatever it may be, has few or none of the attributes which common sense ascribes instinctively to reality. A chair may be a collection of whirling electrons, or an idea in the mind of God: each of these accounts of it may have its merits, but neither conforms at all closely to the suggestions of common sense.

I went on to say that neither physicists nor philosophers have ever given any convincing account of what 'physical reality' is, or of how the physicist passes, from the confused mass of fact or sensation with which he starts, to the construction of the objects which he calls 'real'. Thus we cannot be said to know what the subject matter of physics is; but this need not prevent us from understanding roughly what a physicist is trying to do. It is plain that he is trying to correlate the incoherent body of crude fact confronting him with some definite and orderly scheme of abstract relations, the kind of scheme which he can borrow only from mathematics.

A mathematician, on the other hand, is working with his own mathematical reality.

Of this reality, I take a 'realistic' and not an 'idealistic' view ... This realistic view is much more plausible of mathematical than of physical reality, because mathematical objects are so much more what they seem. A chair or a star is not in the least like what it seems to be; the more we think of it, the fuzzier its outlines become in the haze of sensation which surrounds it; but '2' or '317' has nothing to do with sensation, and its properties stand out more clearly the more closely we scrutinise it. It may be that modern physics fits best into some framework of idealistic philosophy – I do not believe it, but there are eminent physicists who say so. Pure mathematics, on the other hand, seems to me a rock on which all idealism flounders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but because it is so, because mathematical reality is built that way.

Although this is eloquently put, many other mathematicians have disagreed. To begin to derive some insight into this difficult question, we should examine the nature of mathematics itself more carefully. If we can see exactly why maths differs from the sciences and other disciplines, then we might be able to make some progress.

The nature of mathematics

Philosophy is a game with objectives and no rules. Mathematics is a game with rules and no objectives.

Anon

What is the average of this set of numbers?

1, 1, 1, 1, 3, 4, 4, 4, 5, 5, 1027

The answer, of course, depends on what we mean by 'average'.

- If we mean 'add them up and divide by the number of items' then the answer is 96.
- If we mean 'the most common number in the list' then the answer is 1.
- If we mean 'the number in the middle of the list' then the answer is 4.

So which is correct? Which is true? Mathematicians use all three meanings – they are called the 'mean', 'mode' and 'median' respectively. This may seem like a trivial matter, but it is actually central to the nature of mathematics. It doesn't really matter which definition of 'average' we use but, once we have decided, there is only one correct answer. Now some definitions may be more useful than others – we may have good reasons for picking one definition over another but until we have decided where to start from we can make no mathematical progress.

Mathematics always works this way. We start from certain assumptions and definitions, which we call **axioms**. We take these without question. From these we can use the rules of logic to work out problems and to find other results, which we call **theorems** and which are known with complete certainty. Beyond the school level, proving theorems is largely what mathematics is all about.

As a simple example of the mathematical process, imagine a child learning about odd and even numbers. She starts by being given a list of odd numbers 1, 3, 5, 7, 9, 11, 13 ... and even numbers 2, 4, 6, 8, 10, 12, 14 ... Her first job is to tell whether other numbers, say 34, 77 and 66 are odd or even. Once competent in this, she may notice a few patterns. It seems that adding 1 to an odd number gives an even number, and adding 1 to an even number always gives an odd number. It also seems that it's only the last digit that makes a number odd or even; the other digits don't make any difference. She may also spot that adding two odd numbers always gives an even number, or that multiplying two even numbers always gives another even number.

Well, we need to be careful here. We are using the word 'always' a little hastily. After all, there is an infinity of numbers, and the child has only experimented with a few dozen. With several examples, she may have a pretty good idea that the pattern always holds, but this isn't enough. A scientist or historian may have to rest content with 'sufficient evidence' (whatever that may mean), but the mathematician can go one step further. In this case, we can easily prove that the patterns are true for all odd and even numbers. Two examples are given below. They may seem a little pedantic, but the techniques can be generalised to more difficult cases, and they allow us to arrive at certain knowledge. Given the axioms, it is impossible to doubt the conclusion of these steps.

Axioms

- An odd number is a number which can be written as $2n + 1$, where n is a whole number.
- An even number is a number which can be written as $2n$, where n is a whole number.
- The usual laws of arithmetic apply.

Check that the definitions of odd and even numbers make sense to you. Experiment with them until you are happy that they are correct definitions. Let n be 5, 7, 50, 100 or anything else you like and see what you get in the two definitions (this 'playing' is a vital part of maths).

Theorem 1: An odd number and an even number add together to give an odd number.

Proof: Let the odd number be o and the even number be e.

Then $o = 2n + 1$ and $e = 2m$ for some whole numbers n and m, by definition.

$$\begin{aligned} \text{So } o + e &= 2n + 1 + 2m \\ &= 2m + 2n + 1 \\ &= 2(m + n) + 1 \\ &= 2p + 1 \text{ where } p \text{ is a whole number} \end{aligned}$$

but this is of the form $2n + 1$ and hence odd. QED

Theorem 2: Two odd numbers add together to give an even number.

Proof: Let the odd numbers be a and b.

Then $a = 2n + 1$ and $b = 2m + 1$ for some whole numbers n and m.

$$\begin{aligned} \text{So } a + b &= 2n + 1 + 2m + 1 \\ &= 2m + 2n + 2 \\ &= 2(m + n + 1) \\ &= 2p \text{ where } p \text{ is a whole number} \end{aligned}$$

but this is of the form $2n$ and hence even. QED

These are hopefully straightforward examples, and the results hardly need formal proof – we 'knew' they were true beforehand. However, more complex problems are only really understood once the proofs have been developed, or counter-examples found, and formal proof is what mathematics (beyond the school level) is all about. New mathematics happens in precisely this way – there is a result which may be believed to be true, but not accepted until the proof has been found. The proof is everything, and this is the defining characteristic of mathematics.

A Prove the following theorems:
Theorem 3: Trebling an even number results in another even number.
Theorem 4: Two even numbers multiplied together give an even number.
Theorem 5: Trebling an odd number gives another odd number.
Theorem 6: An odd number and an even number multiplied together give an even number.
Theorem 7: Two odd numbers multiplied together give an odd number.

You have probably noticed that the claim was made for 'certainty' but not for 'truth'. This is an important distinction and we can see that the 'truth' of mathematics will depend on the axioms. We may apply all the logic we want, but if the axioms we start with aren't any good then we won't get anywhere (this is the 'garbage-in, garbage-out' principle). In the example above, we took as axiomatic, 'An odd number is a number which can be written as $2n + 1$, where n is a whole number.' Is this true? In a way, it is hard to see how it could be true or false – there are numbers of the form $2n + 1$, and we can call them odd if we want to. All we are doing is giving certain things certain names. Does a pentagon really have five sides? Well, yes, but only because we define pentagons to be five-sided shapes! If we want to take these as certain truths then it has to be said they seem rather empty of content.

The plus side of this method is that if we accept the axioms as true then we do not have to worry about the truth of the conclusion – if we have done our maths right then the conclusion is guaranteed. In this sense, all maths is implicit in the axioms. H. A. Simon writes:

All mathematics exhibits in its conclusions only what is already implicit in its premises. Hence all mathematical derivation can be viewed simply as change in representation, making evident what was previously true, but obscure. This view can be extended to all of problem solving – solving a problem simply means representing it so as to make the solution transparent.

A What is the relationship between truth and mathematics? Why has it been said that maths is a **formal game** or a **closed system**?

B Is all mathematics really just a change in representation? Might the same be said of any other forms of knowledge, or even all forms of knowledge?

So the relationship between truth and mathematics is a difficult one. For our purposes, we can merely note that maths may be certain, but it is far from obvious that it is true, in the usual sense of the word, because the truth of the axioms is not clear. This might suggest that if we could somehow find more definite axioms, the mathematical method of logical deduction might provide a wonderful method for acquiring knowledge. All we need to do is find some certain axioms from which to start. René Descartes had the same idea several hundred years ago. This form of approach to knowledge is called **rationalism**, and is still hugely influential in many spheres of intellectual life today.

Maths as a creative art

Mathematics, rightly viewed possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture

Bertrand Russell

So far we have concentrated on the logical side of maths. Certainly, logic plays a very central role, but there is more to maths than that and, in particular, there is a great deal of creativity and imagination. You may not have seen much evidence of that in the proofs of theorems 1 and 2 (page 57), where each step followed logically and there seemed to be little room for originality or inspiration. But we can easily find problems where a 'logical' approach (what does that mean anyway?) doesn't get us very far.

Recall that we say that a positive whole number is a *prime* number if it has exactly two factors. That is,

2 is prime because $2 = 1 \times 2$ so 1 and 2 are the only factors of 2

and 17 is prime because $17 = 1 \times 17$ so 1 and 17 are the only factors of 17

but 21 is not prime because although $21 = 1 \times 21$ (so 1 and 21 are factors of 21) we also have $21 = 7 \times 3$ (so 1 and 21 are not the *only* factors).

With this in mind, we can see that the first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 . . .

Now these numbers prove to be very interesting to mathematicians, because they are to arithmetic what the elements are to chemistry. In chemistry, you study the elements so that you understand how more complex substances (which are made up of elements) behave. So, too, in maths we can study prime numbers with a view towards generating insights which work for 'more complex' numbers. So, let us ask ourselves a few questions about primes:

- Are there any more even prime numbers after 2?
- How many prime numbers are there?
- Do the gaps between the primes keep getting bigger?

Can you answer these questions? Can you prove them? Things are getting a little more complex here. There is no immediately obvious way to start trying to prove these – you may have a pretty good idea about the answer (your intuition may be quite well developed) but the formal, logical proof is far from straightforward. And, of course, until the proof is there, mathematicians are going to look at intuition with a fairly sceptical eye. And what about these questions:

- Is there a prime between n and $2n$ for any value of n?
- Is there a prime number between successive square numbers?
- How many prime numbers are exactly 1 more than a square number?
- How many pairs of prime numbers are there which differ by 2 (for example, 11 and 13 or 10,006,427 and 10,006,429)?
- Is every even number greater than 2 the sum of two prime numbers?

How to start these proofs? It is not at all obvious; the definitions of 'prime', 'square' and 'even' do not really seem to help; and there is no clear way to begin. In fact, if you can answer, and prove your answer, to either of the last two questions then you will be a very, very famous mathematician. (The last question was set by Goldbach, 1690–1764, who notoriously conjectured that there is an infinite number of such pairs. It remains one of the outstanding problems of number theory.)

Of course, we are less interested in the actual problems themselves than we are in what they tell us about the nature of the discipline, but you must not imagine that your experience of some of these problems is all that different to that of the professional. You may both look at a problem, understand what it is that you want to do, but be unable to see a way of doing it. The difference is that in school maths you can ask your teacher or look up a text, but for the professional, there may be no one to ask and no books to consult. He is on his own, and he has to come up with something new, something that nobody else has ever thought of.

A Have you ever solved a maths problem when no one had told you a method or a way of doing it? Have you ever found a solution all by yourself?
B How is this process similar to or different from the scientist, the historian, the novelist or the musician at work?
C So how do mathematicians do it? How do they come up with new ideas?

Of course, part C of this question is impossible to answer. If we could answer it, then we would be back to the stage of reducing maths to a recipe, and mathematicians would merely be following the instructions. We can point to factors that may help creativity – relevant experience, love of subject or whatever – but these are not, in themselves, enough. Plenty of people may be trying to create (discover?) something new, and they may all have 'the right background', but only one actually manages it. The key to their insight is often as obscure to the mathematician as it is to anyone else. Creativity cannot be quantified easily. Recognising this is perhaps the key to understanding why some mathematicians see themselves as artists, and certainly key to understanding why some maths is considered 'great' and other maths not. Great maths, like any great art, does not follow well-trodden paths, nor does it apply tried and tested techniques. Instead it does something genuinely new, deep or profound. Like any great art, great maths is beautiful. The idea of beauty in maths has never been better expressed than by G. H. Hardy:

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. A painter makes patterns with shapes and colours, a poet with words . . . A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less than words.

The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place for ugly mathematics.

He goes on to say:

I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or for ill, the least difference to the amenity of the world. Judged by all practical standards, the value of my mathematical life is nil. I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value. The case for my life ... is that I have added something to knowledge ... and that this has a value which differs in degree only, and not in kind, from the creations of the great mathematicians, or any of the other artists, great or small, who have left some kind of memorial behind them.

A Is mathematical creativity the same as other types of creativity? If not, what are the differences?

B Although Pythagoras' theorem is named after Pythagoras, anyone could have 'found' the theorem. Contrast this to literature. Could anyone else have written Shakespeare's or Dostoevsky's works? How about music, poetry or architecture?

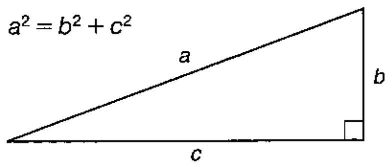
C It is unfortunate that so much mathematics remains inaccessible to so many. However, we can see where aesthetic appeal comes in from a few simple examples. Consider the following mathematical statements:

$\sqrt{16} = \pm 4$ $\quad$ $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

$1 + 3 = 4$ $\quad$ $\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times \dots}{1 \times 1 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \dots}$

$2764/23 \approx 116.26$ $\quad$ $5 + 9 = 2$

$a^2 = b^2 + c^2$



$e^{i\pi} + 1 = 0$

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots = \frac{n}{2n}$

$$\begin{aligned} 35^2 - 25^2 &= (35 + 25)(35 - 25) \\ &= 60 \times 10 \\ &= 600 \end{aligned}$$

Could any of these statements be considered beautiful in any way? You may find it helpful to consider notions of brevity, simplicity, truth, utility, elegance and surprise.

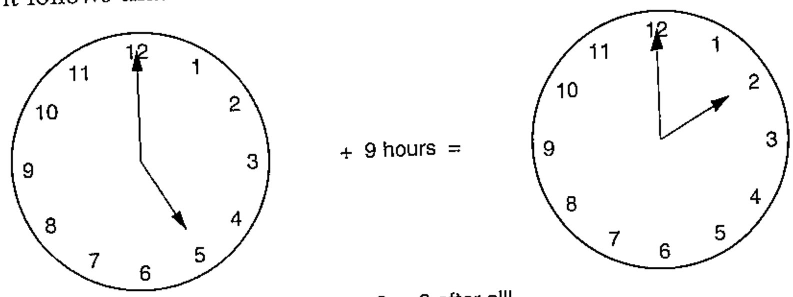
D Is there any difference between the beauty in maths and the beauty in, say, music?

A little more about axioms

The truth of a theorem depends on the truth of the axioms, but axioms cannot be true or false.

Anon

You have seen that the choice of axioms is central to mathematics. So how do we choose our axioms? It may seem at first that we have no choice over our axioms, at least in certain fields. After all, isn't it true that $5 + 9 = 14$ no matter what our axioms are? Well, in fact no! We can easily change our axioms so that $5 + 9 = 2$; all you have to do is think about clock arithmetic. Moving the hour hand five hours ahead followed by nine hours ahead is the same as moving it two hours ahead. We then generate a whole lot of other 'truths', such as $11 + 1 = 0$, $7 \times 2 = 2$ and so on. These are mathematically correct in the axiomatic system described. We can choose that system and then it follows that $5 + 9 = 14$ will no longer be true!



So 5 + 9 = 2 after all!

So why do we use the number system that we do? The answer is simply that we use it when it is convenient to do so. In the physical world, when we add five things to nine things, we end up with fourteen things, so we say $5 + 9 = 14$. But on a clock face that doesn't work, so we use another system. Similarly, you may see chapters in a book numbered 1.1, 1.2, 1.3 ... all the way up to 1.9 and then 1.10, 1.11, 1.12. This is incorrect in our normal numbering system, but it is convenient to use in this context. In quantum mechanics, physicists use a system whereby it is possible for one particle and another particle to add up to no particles, simply because it works. So this is the first way we choose our axioms – we see what is useful.

Of course, after reading the last section you know that not all mathematicians are mathematicians because they want to do something useful! They are far more interested in finding insights, elegance and surprises. This affects the choice of axiom, too. Sometimes an axiom can be chosen which seems at odds with anything useful at all. For example, it is possible to construct versions of mathematics where the order of multiplication is important, that is, where $a \times b$ is not the same as $b \times a$. Now our ordinary numbers don't work that way, but we can get some very interesting maths like this. The surprising thing is that, if we construct this maths, it often turns out, later on, that a use can be found for it, even though it was designed purely with aesthetic properties in mind. This seems to indicate a profound truth about the Universe, and reminds us of Hardy's comment: '*Beauty is the first test: there is no permanent place for ugly mathematics*.' It would be a wonderful thing indeed, if, as the physicists Dirac and Einstein hoped, the mathematics describing the world is, at a deep level, profoundly satisfying aesthetically. Perhaps the two methods of choosing axioms, utility and elegance, are not so different after all.

So we are perfectly at liberty to choose any axioms we want, and to work with them to see what develops. Some sets of axioms (the vast majority) will be sterile and uninteresting. Others will generate rich areas with seemingly endless practical and/or aesthetic possibilities. Versions of mathematics that are at first sight bizarre are easy to dismiss, but like the genius artist who starts a new style of painting or music, the genius mathematician is the one who chooses the axioms nobody else even suspected.

A In clock arithmetic, calculate the following:

5 + 8	3 × 2
2 − 4	5 × 10
8 − 12	11 + 12

B What are the right axioms for arithmetic? Is everyday arithmetic 'true'?

You may think that this axiomatic approach must be very tedious; must even professors start right from the basics and prove everything absolutely rigorously? The answer is no; in practice once someone has proven a result to the satisfaction of the mathematical community then that result can be used without further proof. So, for example, you can use Pythagoras' theorem, or the cosine rule, or the formula for the area of a circle, quite happily *as they have been proven already*. In theory you could go right back to the axioms and prove them again from scratch, but there wouldn't be much point in doing so – even though it must, in theory, be possible to do exactly that. When we use these already-established results as the basis of work we tend to refer to them as theorems rather than axioms, reserving the term *axiom* for the very basic results, but in practice they play the same role; we use them without proof. The difference is that 'real' axioms *cannot* be proven as they are the original starting points; whereas *theorems* have already been proven and so can be used as more advanced starting points. Mathematicians have a huge body of theorems on which they draw; it is unheard of these days for them to go back to the axioms, not least because it would take so long to do so. Mathematicians are human after all!

Considering this axiomatic method may offer a resolution of the discovery/invention dilemma. We are free to invent whatever axioms we choose, and we then discover the consequences of our choices. What we are saying here is, in a sense, blindingly obvious – that we must start our argument from somewhere, and even if we don't like the starting points we can develop an argument from them. Any lawyer knows this!

It turns out, however, that there are some very surprising consequences of adopting such an approach. In adopting an axiom, mathematicians are committing themselves to its logical consequences, so they are careful to choose 'safe' axioms. The so-called 'axiom of choice', for example, first noted by Ernst Zermelo in 1904, states, roughly, that if you have a collection of non-empty sets, then you can make a new set by choosing elements of the original sets. So, for example, if you have 15 rugby teams, then you can create a new team of 15 by choosing one player from each of the 15 teams. This seems almost too obvious to bother stating – and it turns out that if you reject it then you have to reject a lot of standard maths with very practical applications in the real world. To lose this axiom would hobble maths. So far so good – why would you reject such an intuitively obvious axiom? Why not accept the axiom of choice and all its consequences? This is where the details get very technical but, to cut a long story short, the axiom of choice has some consequences which are so counter-intuitive that some mathematicians have actually rejected it – and when you consider how 'obvious' it is, this is extremely surprising. The trouble is that if you accept the axiom you have to accept its consequences, and one of them goes by the name of the 'Banach–Tarski paradox', after its discoverers (inventors?). This states that you can take a mathematical sphere the size of a tennis ball, cut it up into little pieces, and simply by re-arranging these pieces *without changing their size*, make a sphere the size of the Earth (an internet search will give you far more details if you want them)! For many, this is just too ridiculous to accept; but, hard as it may be to believe, it is the consequence of an 'obvious' axiom.

A Give an example of what it means to create a new set from a collection of non-empty sets. Do you think it is always possible to do this?

B Do you think it is possible to cut and rearrange a mathematical sphere and obtain a bigger one?

C How would you deal with the fact that agreeing with the first question above commits you to agreeing with the second? Which of your answers would you change?

D Why do you think very, very few mathematicians are happy with saying 'Yes' to the first question but 'No' to the second?

Many mathematicians are not too perturbed by these extremely odd theorems – after all, they apply to the world of mathematics, not the physical world. You might think that the axiomatic approach is then a good way to proceed – and this was indeed the way that mathematics did proceed last century, when the possibilities for this approach seemed enormously exciting. The German mathematician David Hilbert started a search for the perfect mathematical tool – a method of telling for sure whether a theorem could be deduced from the axioms or not. He wanted to find a step-by-step recipe which would determine mechanically whether or not any theorem was true or false in the given axiomatic system. Recalling that mathematics is about proving theorems from axioms, and that the theorems follow by the rigid application of logic to the axioms, this does not seem like too much to ask; we want an algorithm (or computer program) into which we can feed the axioms, and the suggested theorem, and then be told if the theorem is correct. All we need to do is find a way to formalise the process of logical deduction into a set of formal rules. This would then provide an incredible shortcut to the mathematical process. In the early parts of the last century, this seemed very exciting.

But alas, this dream was proven impossible in 1931 by the Austrian Kurt Gödel, at the remarkably young age of 25. In two breathtakingly ingenious theorems he proved that Hilbert's dream was impossible; that in all interesting mathematical systems **there will always be mathematical theorems which are true, but which cannot be proven right or wrong from the axioms** no matter how clever or inventive we are.

Gödel's proofs may not sound particularly revolutionary at first, but some of the consequences of this innocent-sounding statement are still hotly debated and it is not an exaggeration to say that the two theorems permanently destroyed a dream of mathematicians just at the time when they seemed to be on the verge of providing us with a complete picture of the mathematical universe. For mathematicians, the consequences are either depressing or delightful, depending on their point of view. The pessimists lament that mathematics can never be completely reduced to a set of rules which can be rigidly applied and guaranteed to determine truth. The optimists rejoice that the grand game will never end, and that there will always be a place for human ingenuity.

If Gödel's results applied only to mathematics they would be of limited interest, but they may well extend further. It has been argued that, since much physics is based on mathematics, if maths is incomplete in principle then so is physics. This means that there are true scientific results which we will never be able to establish. This might mean that we can never reach the end of science – that certain things will be forever beyond us. Gödel's ideas may also prove that humans will always have more powers of logical insight than computers! The details of this argument are too complex to discuss fully, but briefly, one controversial interpretation of it is that Gödel proved that given *any* complex computer program a human mathematician could always find a mathematical truth, which the program could not decide was true or false. In addition, the human could also prove that this statement is true! Re-read these last statements; the implications may be vitally important for our view of ourselves as humans. If humans can do something that no computer can do, then this might mean that there is something about human intelligence that can never be attained by any computer, even in principle. If this controversial interpretation is correct, it has dramatic implications for scientific research into computing techniques. It might prove, finally, that it is impossible to have a computer which can think like us!

Many thinkers feel sheer astonishment that a purely logical result can offer such insight into human cognitive processes, though the insight is hotly disputed. On a more general philosophical note, some have interpreted Gödel as sounding a death-knell for the whole possibility of certainty, arguing that if complete certainty cannot be found in mathematics, of all places, then it cannot be found anywhere at all. To consider this more carefully, you will have to look in detail at precisely when Gödel's results hold, and precisely what they say. For our purposes, we simply note that, even if we could apply the mathematical method to other systems of knowledge, we would by no means have the perfect truth-generating machine. We can see that no such thing exists, even in the world of mathematics.

A Do you think it is a shame or a great thing that mathematics cannot be axiomatised?

B Do you like the controversial implications of Gödel's theorem?

Maths as a human endeavour

More vigour, less rigour

Nietzsche

We have so far looked at mathematics as a structure, and examined the model of a deductive axiomatic system heavily focused on formal proofs. While this is a commonly held view, and while proofs are the 'content' of the subject, it does overlook the whole process of actually doing mathematics and the fact that it is done by humans – at least for the most part!

Once we identify this gap, some obvious points emerge. Firstly, there is the obvious fact that people make mistakes! We may think we have a result that is 'certain', but that may simply be because we have made an error, and the history of mathematics is littered with false 'proofs'. Perhaps the most famous example of this is the German mathematician David Hilbert's 21st problem; last century he gave a list of challenges and the best minds in the mathematical community set out to solve them. The 21st problem in the list was proven in 1908, and according to a simple model of mathematics you might think this was the end of the matter – but in fact a counter-example to the theorem was found in 1989, and only then did mathematicians discover that the proof was incorrect! Eric Bell has gone so far as to say that '*Experience has taught most mathematicians that much that looks solid and satisfactory to one mathematical generation stands a fair chance of dissolving into cobwebs under the steadier scrutiny of the next*' and one wonders how many other mathematical 'truths' are, in fact, false.

A further and perhaps more important problem is that there are fads and fashions in mathematics, just like in any other human endeavour, and so the very standards of mathematics are open to change – contrary to the 'eternal truth' school of thought. Recently the Four Colour Theorem was controversially proven by an ingeniously programmed computer; and while the programmers obviously knew what they were doing, they did not actually do the proof. Some mathematicians at the time did not accept this as a valid proof – but this view is increasingly rare, and we are currently seeing huge growth in 'experimental maths' that relies far more on inductive results from number-crunching machines than on pure deduction. The important point is that if standards of mathematics are open to change then the best mathematicians can do is to say that 'it seems to work for us, now'. In this respect perhaps there is not such a clear difference between mathematics and the sciences. Mathematician Raymond Wilder has claimed that *'we do not possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it'*. This is hotly contested by other mathematicians, but if it is the case then the lofty claims of the 'queen of the sciences' seem rather over-inflated, and are perhaps more about the desires of the mathematicians than the reality. G. H. Hardy put it strongly:

There is, strictly, no such thing as mathematical proof... we can, in the last analysis, do nothing but point... proofs are what... I call gas – rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils.

Once we start thinking about the psychology of the subject then we are alerted to the fact that to focus solely on the proof is to miss the mathematician's struggle, his adventure. Imre Lakatos puts it well: *'The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility.'*

Of course being sceptical about the methodology we have outlined in this chapter does not require that we resort to total scepticism as to the truth of the *results*. Most mathematicians would claim that theorems are true or false independent of our knowledge of them. Perhaps what evolves is not maths but our knowledge of it – which is surprisingly close to what many historians would say about their discipline – and perhaps what Percy Bridgeman was thinking of when he said *'It is the merest truism that mathematics is a human invention.'*

There is no universally accepted body of mathematics

Morris Klein

Whatever your view of these ideas, it is clear that, contrary to the popular image of maths as right or wrong, black or white, the subject is deeply controversial even to professionals.

Where do we go from here?

In our quest for truth, we looked to mathematics to provide certainty, and to a certain extent we have been successful, but perhaps not as successful as we might have hoped. We have learned that mathematical reasoning based on assumed axioms can generate certain, proven knowledge and, what is more, there even seems to be the possibility of an aesthetic element. Despite Gödel's theorems, this seems to be very promising, and we are immediately led to ask if the mathematical method can be generalised to things other than mathematical objects. If so, then perhaps we have made a significant step in our quest for truth. Traditionally, the application of mathematical principles (logic) to the world has been called rationalism, and it is the subject of the next chapter.

Further reading

It is difficult for the non-specialist to get to grips with much of the mathematical literature, but G. H. Hardy's *A Mathematician's Apology* (Cambridge University Press, 1940 repr. 1994) is a brilliant and engaging description for the lay person. If you would like to get a first-hand, totally non-algebraic experience of mathematical imagination, then Edwin Abbott's classic *Flatland* (Penguin, 1952) and its more readable descendent, Rudy Rucker's *The Fourth Dimension (and how to get there)* (Rider and Company, 1985) are unsurpassed for expanding conceptions of mathematics. Two very readable accounts of humans at the centre of mathematics are David Blatner's *The Joy of Pi* (Walker & Co., 1999) and Simon Singh's *Fermat's Enigma* (Walker & Co., 1997).

Getting slightly more technical, an outstanding description of what mathematicians actually do can be found in Philip Davis and Reuben Hersh's *The Mathematical Experience* (Houghton Mifflin, 2000). The creative and very human side of the notion of proof is brilliantly explored in play form in Imre Lakatos' *Proofs and Refutations* (Cambridge University Press, 1977). If you want to follow up the ideas behind the Banach–Tarski theorem and similar issues then Morris Kline's difficult but fabulous *Mathematics: The Loss of Certainty* (Oxford University Press, 1982) is well worth the investment in time it will take. A more accessible, rich, wide-ranging and funny account of Gödel's works (linking maths, music and art) can be found in Douglas Hofstadter's *Gödel, Escher and Bach* (Vintage, 1989) – which is more an intellectual experience than a book. The same ground is also covered in the excellent Ernest Nagel, James R. Newman and Douglas R. Hofstadter's *Gödel's Proof* (New York University Press, 2001). For two rather lighter but equally worthwhile books, try John Allen Paulo's *Mathematics and Humour* (University of Chicago Press, 1980), which is a short and funny book, or, as previously mentioned, David Blatner's *The Joy of Pi* (Walker & Co., 1999).

IT AIN'T WHAT YOU PROVE, IT'S THE WAY THAT YOU PROVE IT

A play by Chris Binge.

Act 1: Lesson 1

Teacher: Good afternoon class. For homework I asked you to investigate triangles and to try to find some of their properties. Can anyone tell me what they have discovered?

Alpha: Yes. I have found that the angles of a triangle always add up to 180°.

Teacher: Perhaps you could explain how you came to this conclusion.

Alpha: Well, I drew a great many triangles of varying shapes and sizes and found that in nearly every case the angle sum was 180°.

Beta: Just a moment, did I hear you say 'nearly' every case?

Alpha: Yes – I admit there were a few that seemed to come to 181° or even 179°.

Beta: So your result should say that 'The angles of a triangle nearly always add up to 180°.'

Alpha: No, the evidence was so strong that I can explain the few that didn't by inaccuracies of measurement.

Beta: What you are trying to say is that you cling to your hypothesis despite evidence to the contrary. These are clearly counter-examples to your theory and it is most unmathematical to dismiss them so quickly.

Alpha: There is always experimental error when measurement is involved – errors must be expected, not considered as counter-examples.

Beta: Teacher I protest. Alpha is using language that is more at home in a science laboratory where vague concepts such as 'strength of evidence' and 'experimental error' may be good enough, but this is a maths class. We are concerned with exactness and absolute truth.

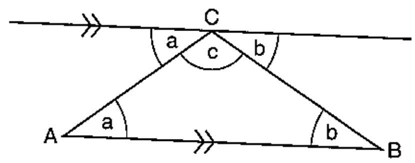
Alpha: Even if I remeasured my triangles more accurately and got 180° every time, I expect you are such a sceptic that you would always say there may be a counter-example I haven't yet found.

Beta: For once you are absolutely correct. No amount of so called 'evidence' will convince me that your hypothesis, however likely, must be true. You are using an *inductive* argument which I cannot accept. I will only believe that when I have a vigorous *deductive* proof that it is the case.

Teacher: I am sure we are all agreed that such a proof would be desirable. Can anybody provide one?

Gamma: Yes. I have a proof that will satisfy Beta. May I demonstrate?

Teacher: Please do.



Gamma: You can see the triangle ABC. It contains angles of size a, b, and c. I have drawn a line passing through C which is parallel to AB. Due to the well-known properties of parallel lines, the angles at point C are also a and b as I have indicated. So now a, b and c are on a straight line, so $a + b + c = 180°$. So Alpha's theorem is proven since this process will work for all triangles.

Teacher: Are there any questions about Gamma's proof, or does this satisfy even Beta?

Delta: Just one small point. You have asserted a 'well-known' result about parallel lines. Could you just prove it for me please.

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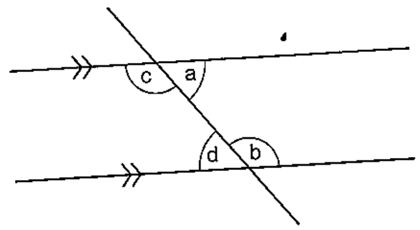
Gamma: OK ... it's due to this property. Since (*pointing*) a + b = 180°, and b + d = 180° then a = d.



Delta: Ah yes, but just one further question, why do a + b = 180°?

Gamma: Well clearly a + c = 180° due to the definition of 180° as the angle (*pointing*) on a straight line. Similarly d + b = 180°. Now, that means a + b + c + d = 360°. Clearly a + b must be equal to c + d otherwise the lines would not be parallel hence a + b = 180°.

Delta: I see. Are you sure there are not other hidden assumptions in your proof?

Gamma: Er yes (*tentatively*).

Delta: In which case may I suggest a couple. Firstly you have assumed that it is always possible to draw a parallel line through a given point. Secondly, you have assumed that it is possible to draw only one such line, that is the one with the angle properties you desire. Can you prove these?

Gamma: You are going to question everything aren't you? Look, a proof is merely an argument from what we already know to be true to a new result. In any proof we must start from assumptions. If you continually question the assumptions we will never be able to reach a new truth. If I use the term 'straight line' there is no point in asking me to prove that it is straight. The same is true with parallel lines. What you are doing is asking for a proof that parallel lines are in fact parallel. All I am saying is, *if* we start with a straight line, *then* we can deduce certain things. I am not, quite frankly, interested in arguing whether or not it is *really* straight. I am assuming it is – if it isn't then we are talking about a different problem.

Phi: To save all this fuss, why not build the angle property into the definition of a triangle and define a triangle as a shape whose angles add up to 180°?

Gamma: You are being facetious. We define a triangle in terms of a few basic concepts, and from these concepts we prove its properties.

Phi: Perhaps you could give us such a definition.

Gamma: Happily. A triangle is a shape formed by joining three points with three straight lines.

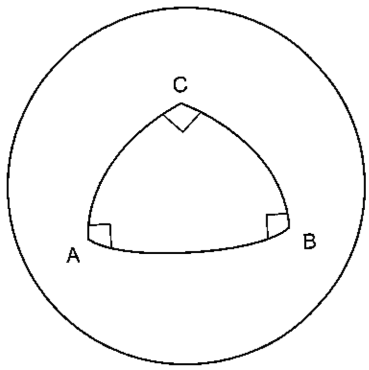
Phi: Now perhaps you will define a straight line.

Gamma: (*wearily*) A straight line is the shortest path that you can draw between two points.

Phi: I will not go on to ask for a definition of points because I already have a counter-example to your theorem, based entirely on the definitions you have given. I have found a triangle whose angles add up to 270°.

Teacher: Please demonstrate.

Phi: (*holds up football*) As you can see, this line gives the shortest path between A and B, the same for BC and CA. All angles are right angles, hence the total is 270°.



Delta: He's right, you cannot deny that this triangle fits your definitions, but it clearly doesn't follow the result of the theorem.

Gamma: This is ridiculous, that is not a triangle. A triangle is a shape drawn on a flat plane, not on a curved surface.

Delta: I'm sorry Gamma but you never put that in your definition. By your definition there are three straight lines joining three points hence this is a triangle, hence a counter-example.

Gamma: The concept of a triangle being a plane figure is implicit in the definition even if it's not explicit.

Alpha: Even I have to disagree here. If I were to go from Singapore to Tokyo to Sydney and back to Singapore by the shortest routes you would all call my path triangular, yet as Phi has shown the angles do not add up to 180°.

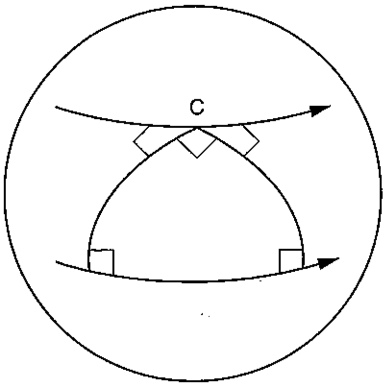
Gamma: Clearly I must make the implicit explicit. I will rephrase the theorem. The angles of a triangle in a plane surface add up to 180°.

Teacher: Before we discuss this any further may I draw your attention to the proof of the theorem? We were happy with the proof and surprised by the counter-example. Should we not examine the proof to see where it breaks down? Then perhaps we will see if there are any other implicit assumptions that must be made explicit.

Alpha: It is the parallel line bit that breaks down.

Phi: I never liked that bit.

Alpha: If we follow the proof like before then you can see that you get three right angles on the line at C! But that's impossible! So the proof doesn't make sense in this case – and you assumed that it would.



Phi: Mmmm ... yes. And you know I'm not even sure that the two lines are parallel. Can you be certain that parallel lines can be drawn on a plane? I suggest that any two lines you draw will meet somewhere, if we have a long enough piece of paper. I challenge you to provide an infinitely long piece of paper to prove me wrong.

Alpha: Any lines I draw will be subject to error in measurement and inaccuracy in construction.

Beta: Oh don't start that again, we have had enough science for one day. There is a better way round the problem.

Alpha: Which is?

Beta: Which is to state clearly all assumptions that we are going to call on, and make our definitions subject to those assumptions. I shall call the assumptions 'axioms' and from then we can deduce 'theorems'.

Phi: But what if your assumptions are false?

Beta: Truth or falsehood doesn't enter into it. We assume our assumptions, obviously. That's why they are called assumptions. Therefore anything that follows from them is true in any world where they hold. If you can't find a world where they hold then it doesn't invalidate the theorems or the argument used to deduce them.

Phi: Let us hear your axioms.

Beta: Certainly.

1) There is one and only one straight line between two points.
2) Any finite straight line can be produced indefinitely.
3) All right angles are equal.
4) A circle can be drawn with any point as centre to pass through a given point.
5) Through any point one and only one line can be drawn parallel to a given line.

Teacher: (*an aside to audience*) The axioms were first suggested by the Greek mathematician Euclid over 2000 years ago. They were accepted as the basis for geometry until the nineteenth century when new systems of axioms were considered and new geometries were explored, including that of the sphere.

Gamma: So if we consider these axioms as our starting point, they define what we might call two-dimensional Euclidian space and it is not necessary or meaningful to question their truth since they are the starting point.

Phi: Surely we should define the terms that we use! We must be able to say what we mean by point and line or the axioms themselves are meaningless.

Delta: No, that would be too restrictive, even if it were possible.

Teacher: I think you should explain that statement – how are definitions restrictive?

Delta: Well the axioms that Beta gave us were envisaged in a flat plane, and our points and lines would be so defined.

Phi: Indeed, it is flat plane geometry we are talking about.

Delta: But if we can find another system which obeys the same again then all the theorems which are true for the flat plane are true for the other system.

Phi: I am a bit worried about the direction in which we seem to be moving. We seem to have lost our grip on reality.

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Teacher: Perhaps you could elaborate on your fears.

Phi: I shall try. When Alpha first suggested the theorem about triangles, he was, quite rightly, criticised for using what one can only call a scientific method. I mean no insult by this. He allowed experimental evidence to guide his thinking and his conclusion was not an accurate deduction from his results. In maths we are not concerned with measurement of angles and the accuracies and inaccuracies that go with it. We are concerned with the theory of angles and triangles, with provable deductions that have a universal truth.

Teacher: Surely that means you must applaud the move towards an axiomatic structure and clearly defined rules of inference.

Phi: Only to a certain extent. It seems to me that we have gone too far. By suggesting that our initial concepts need have no definitions we have lost any relevance that our results may have to a real situation.

Teacher: Alpha, are you in broad agreement with Phi?

Alpha: I agree with him about going too far and leaving reality behind. It seems to me that maths has no value unless it informs us more about the world we live in and Delta's deduction from axioms and undefined terms seems to be little more than a game. I do however still defend the experimental approach as a starting point, because unless I had found the hypothesis by drawing then we would have had nothing to prove and hence no work to do. The correct procedure must be to find a result by experiment and then, using agreed definitions, we must prove the result true. The important thing is that the definitions characterise the objects of discussion.

Delta: I am sorry, but I disagree. The picturing of any reality is irrelevant, and to look for such a picture is not the purpose of mathematics. The job of a mathematician is to set up axiomatic systems and to deduce from them theorems. Our conceptions of the real are not fixed, they vary from person to person and they change, within each person from time to time. One only has to look at the confusion caused when Einstein asked scientists to drop their Newtonian ideas of physics or the continuing debate over quantum theory and wave theory to see how any supposed picture of reality is inadequate. Whether or not an axiomatic system is of any value to scientists does not affect its validity as a piece of mathematics. We are not concerned with perceptions of an external reality, we are concerned with objects created by the mind, and rules we use to govern these objects. As such the objects cannot and should not be defined in terms of the real world, since the real world, or at least our view of it, will change.

Beta: I agree with Delta. I also noticed Alpha's attempt to slander axiomatic systems by calling them games. He is probably so upset at being called a scientist that he wanted to throw a few insults of his own. However, he has failed miserably as I do not consider the word 'game' an insult at all. The game of chess is a very good analogy. In chess the pieces have names and their rules for movement are the axioms. A position is allowable only if it can be reached by using the rules. But the pieces are not defined in terms of anything outside chess. We call a bishop a bishop and a knight a knight but their rules of movement bear no relation to any bishops or knights outside the game of chess (if they did then the phrase 'queen mates with bishop on back row' would have a completely different meaning). No attempt is made to use the game as a picture of reality. The pieces are purely man-made concepts and the game is a formal logical structure. Mathematics is a formal logical structure derived from rules in the same way – and the greatest game of all.

A What is the role of experiment in mathematics? How does this differ from the role of experiment in science?

B Once mathematicians believe that they have found a result (or theorem), what is the next step? How does this differ from science?

C What is an axiom? Can an axiom be true or false?

D In your own words, summarise Delta's position on mathematics. How does he differ from Alpha regarding the status of axioms?

E So are the results of mathematics true? What do we mean by mathematical truth?

'The number system is like human life . . .'

An extract from *Miss Smilla's Feeling for Snow* by Peter Høeg.

'I'm afraid of being locked up,' I say.

He puts the crabs in the pot. He lets them boil for no more than five minutes.

In a way I'm relieved that he doesn't say anything, doesn't yell at me. He's the only other person who knows how much we know. It seems necessary to explain my claustrophobia to him.

'Do you know what the foundation of mathematics is?' I ask. 'The foundation of mathematics is numbers. If anyone asked me what makes me truly happy, I would say: numbers. Snow and ice numbers. And do you know why?'

He splits the claws with a nutcracker and pulls out the meat with curved tweezers.

'Because the number system is like human life. First you have the natural numbers. The ones that are whole and positive. The numbers of a small child. But human consciousness expands. The child discovers a sense of longing, and do you know what the mathematical expression is for longing?'

He adds cream and several drops of orange juice to the soup.

'The negative numbers. The formalisation of the feeling that you are missing something. And human consciousness expands and grows even more, and the child discovers the in-between spaces. Between stones, between pieces of moss on the stone, between people. And between numbers. And do you know what that leads to? It leads to fractions. Whole numbers plus fractions produces the rational numbers. And human consciousness doesn't stop there. It wants to go beyond reason. It adds an operation as absurd as the extraction of roots. And produces irrational numbers.'

He warms the French bread in the oven and fills the pepper mill.

'It's a form of madness. Because the irrational numbers are infinite. They can't be written down. They force human consciousness out beyond the limits. And by adding the irrational numbers to the rational numbers you get the real numbers.'

I've stepped out into the middle of the room to have more space. It's rare that you have a chance to explain yourself to a fellow human being. Usually you have to fight for the floor. And this is important to me.

'It doesn't stop. It never stops. Because now, on the spot, we expand the real numbers with imaginary square roots of negative numbers. These are numbers we can't picture, pictures that normal human consciousness cannot comprehend. And when we add the imaginary system to the real numbers, we have the complex number system. The first number system in which it's possible to explain satisfactorily the crystal formation of ice. It's like a vast, open landscape. The horizons. You head towards them and they keep receding. That is Greenland, and that's what I can't be without. That's why I don't want to be locked up.'

I wind up standing in front of him.

'Smilla,' he says, 'can I kiss you?'

A Smilla says, 'the foundation of mathematics is numbers'. Is this really the foundation? Are there any other contenders for the foundation of mathematics?

B Smilla describes some operations as 'absurd', and some numbers as 'madness'. What are her grounds for doing so? Are these reasonable grounds?

C Why does Smilla liken maths to human consciousness? Does this analogy tell you anything? Do any of your experiences suggest anything similar?