7 Mathematics

'Mathematics is neither physical nor mental, it's social.' **Reuben Hersh, 1927–**

'The useful combinations [in mathematics] are precisely the most beautiful.' **Henri Poincaré, 1854–1912**

'Mathematics is the abstract key which turns the lock of the physical universe.' **John Polkinghorne, 1930–**

'Everything that can be counted does not count. Everything that counts cannot be counted.' **Albert Einstein, 1879–1955**

'The mark of a civilized man is the ability to look at a column of numbers and weep.' **Bertrand Russell, 1872–1970**

'The advancement and perfection of mathematics are intimately connected with the prosperity of the state.' **Napoleon Bonaparte, 1769–1821**

'A mathematician is a machine for turning coffee into theorems.' **Paul Erdos, 1913–96**

'Mathematics began when it was discovered that a brace of pheasants and a couple of days have something in common: the number two.' **Bertrand Russell, 1872–1970**

'Math – that most logical of sciences – shows us that the truth can be highly counterintuitive and that sense is hardly common.' **K. C. Cole**

'To speak freely, I am convinced that it [mathematics] is a more powerful instrument of knowledge than any other...' **René Descartes, 1596–1650**

'Instead of having "answers" on a math test, they should just call them "impressions", and if you got a different "impression", so what, can't we all be brothers?' **Jack Handy, 1949–**

'If I could prove by logic that you would die in five minutes, I should be sorry you were going to die, but my sorrow would be very much mitigated by pleasure in the proof.' **G. H. Hardy, 1877–1947, to Bertrand Russell**

'The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious, and there is no rational explanation for it.' **Eugene Wigner, 1902–95**

'In the pure mathematics we contemplate absolute truths which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of their radiant host shall have fallen from heaven.' **Edward Everett, 1794–1865**

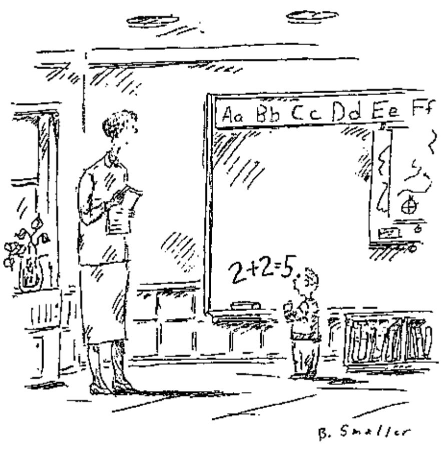
Introduction

Mathematics is a subject that seems to charm and alarm people in equal measure. If someone asks you, 'What are you most certain of in the world?' you might reply '2 + 2 = 4'. Surely no one can doubt that! Mathematics seems to be an island of certainty in a vast ocean of doubt.

At the most general level, we might characterise mathematics as the search for abstract patterns. And such patterns turn up evertwhere. When you think about it, there is something extraordinary about the fact that, for *anything* you care to name, if you take two of that thing and add two more of that thing you end up with four of that thing. Similarly, if you take any circle – no matter how big or small – and divide its circumference by its diameter, you *always* end up with the same number – π (roughly 3.14).

The fact that there seems to be an underlying order in things might explain why mathematics not only seems to give us certainty, but is also of enormous practical value. At the beginning of the scientific revolution, Galileo (1564–1642) said that the book of nature is written in the language of mathematics. If anything, mathematics is even more important than it was in the seventeenth century, and mathematical literacy is a prerequisite for a successful career in almost any branch of science.

The certainty and usefulness of mathematics may help to explain its enduring appeal. The mathematician and philosopher Bertrand Russell (1872–1970) recalled how he began studying geometry at the age of eleven: 'This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world.' Russell's description would be greeted with blank incomprehension in some quarters. For many people, words such as 'love' and 'delicious' simply do not go with the word 'mathematics'. Mathematics may give some a reassuring feeling of certainty, but others find it threatening precisely because it leaves us with no place to hide. If you make a mistake in a maths problem you can be *shown* to be wrong. You can't say it's 'an interesting interpretation', or 'an original way of looking at it', or 'it all depends what you mean by...' You're just wrong!



"Maybe it's not a wrong answer — maybe it's just a different answer."

Figure 7.1

To what extent do you think people's beliefs about the value of mathematics are determined by their ability in the subject?

Mathematical thinking also requires a kind of selective attention to things; for you have to ignore context and operate at a purely abstract level. While some people find the resulting abstractions fascinating, others can find little meaning in them. The American novelist Philip Roth gives an amusing account of a father trying to sharpen the mind of his son, Nathan, by throwing maths problems at him:

> 'Marking Down', he [my father] would say, not unlike a... student announcing the title of a poem. 'A clothing dealer, trying to dispose of an overcoat cut in last year's style, marked it down from the original price of thirty dollars to twenty-four. Failing to make a sale, he reduced the price to nineteen dollars and twenty cents. Again he found no takers, so he tried another price reduction and this time sold it... All right, Nathan, what was the selling price if the last markdown was consistent with the others?' Or, 'Making a Chain.' A lumberjack has six sections of chain, each consisting of four links. If the cost of cutting open a link....' and so on.
>
> The next day... I would day dream in my bed about the clothing dealer and the lumberjack. To whom had the haberdasher finally sold the overcoat? Did the man who bought it realize it was cut in last year's style? If he wore it to a restaurant, would people laugh? And what did 'last year's style' look like anyway? 'Again he found no takers', I would say aloud, finding much to feel melancholy about in that idea. I still remember how charged for me was that word 'takers'. Could it have been the lumberjack with his six sections of chain who, in his rustic innocence, had bought the overcoat cut in last year's style? And why suddenly did he need an overcoat? Invited to a fancy ball? By whom?...
>
> My father... was disheartened to find me intrigued by fantasies and irrelevant details of geography and personality and intention, instead of the simple beauty of the arithmetic solution. He did not think that was intelligent of me and he was right.

The very success of mathematics has sometimes bred a kind of 'imperialism' which says that if you can't express something in mathematical symbols then it has no intellectual value. You might, however, feel that many important things in life escape the abstractions of a formal system.

The mathematical paradigm

A good definition of mathematics is 'the science of rigorous proof'. Although some earlier cultures developed a 'cookbook mathematics' of useful recipes for solving practical problems, the idea of mathematics as the science of proof dates back only as far as the Greeks. The most famous of the Greek mathematicians was Euclid who lived in Alexandria, Egypt, around 300 BCE. He was the first person to organise geometry into a rigorous body of knowledge, and his ideas have had an enduring influence on civilisation. The geometry you study in high school today is basically Euclidean geometry.

The model of reasoning developed by Euclid is known as a **formal system**, and it has three key elements:

axioms
deductive reasoning
theorems.

When you reason formally, you begin with *axioms*, use *deductive reasoning*, and *derive theorems*. The latter can then be used as a basis for reasoning further and deriving more complex theorems.

Axioms

The axioms of a system are its starting points or basic assumptions. At least until the nineteenth century, the axioms of mathematics were considered to be self-evident truths which provided firm foundations for mathematical knowledge. You might want to be awkward and insist that we prove our axioms. But, as we saw in Chapter 5, you can't prove everything. If you tried to, you would get caught in an **infinite regress** – endless chain of reasoning – proving *A* in terms of *B*, and *B* in terms of *C* and so on for ever. We have to start somewhere, and there is surely no better place to start than with what seems to be obvious.

There are four traditional requirements for a set of axioms. They should be consistent, independent, simple and fruitful.

1 *Consistent* If you can deduce both *p* and non-*p* from the same set of axioms they are not consistent. Inconsistency is bad news because, once you allow it into a system, you can prove literally anything.
2 *Independent* For the sake of elegance, you should begin with the smallest possible number of axioms. You should not be able to deduce one of the axioms from the others – for then it is a theorem rather than an axiom.
3 *Simple* Since axioms are accepted without further proof, they ought to be as clear and simple as possible.
4 *Fruitful* A good formal system should enable you to prove as many theorems as possible using the fewest number of axioms.

Starting with a few basic definitions – such as a point is that which has no part, and a line has length but no breadth – Euclid postulated the following five axioms:

1 It shall be possible to draw a straight line joining any two points.
2 A finite straight line may be extended without limit in either direction.
3 It shall be possible to draw a circle with a given centre and through a given point.
4 All right angles are equal to one another.
5 There is just one straight line through a given point which is parallel to a given line.

Deductive reasoning

We discussed deductive reasoning in Chapter 5, and gave as one example of a syllogism:

All human beings are mortal (1)
Socrates is a human being (2)
Therefore Socrates is mortal (3)

(1) and (2), we said, are the **premises** and (3) the **conclusion** of the argument; and if (1) and (2) are true then (3) is *necessarily* true. In mathematics axioms are like premises and theorems are like conclusions.

Theorems

Using his five axioms and deductive reasoning, Euclid derived various simple theorems, such as:

1 Lines perpendicular to the same line are parallel.
2 Two straight lines do not enclose an area.
3 The sum of the angles of a triangle is 180 degrees.
4 The angles on a straight line sum to 180 degrees.

Such simple theorems can then be used to construct more complex proofs. Consider Figure 7.2. You are told that angle *a* plus angle *c* equals 180 degrees, and you are then asked to prove that angle *b* equals angle *c*.

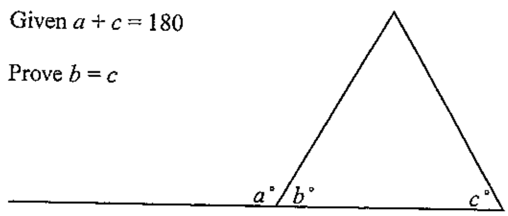
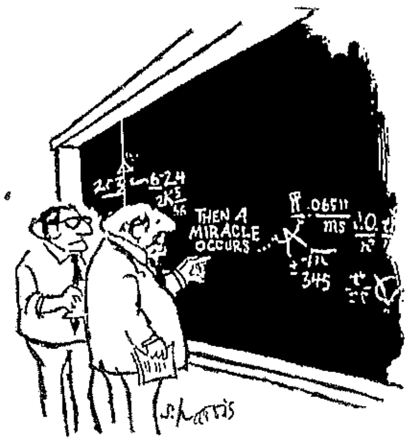


Figure 7.2

Here is a proof:

1		$a + c = 180$	given
2	and	$a + b = 180$	angles on a straight line (theorem 4 above)
3	therefore	$a + c = a + b$	by substitution
4	therefore	$b = c$	QED

One of the attractive things about this proof is its generality. Whatever the size of angle *a* – be it 102 degrees or 172 degrees – if we know that angle *a* plus angle *c* equals 180 degrees, then angle *b* *must* equal angle *c*.



"I think you should be more explicit here in step two."

Figure 7.3

Proofs and conjectures

We have seen that a formal system begins with axioms and uses deductive reasoning to prove theorems. To clarify what is meant by a 'proof' in the strict mathematical sense of the word, we can compare a proof with a conjecture. In a *proof* a theorem is shown to follow logically from the relevant axioms. A conjecture, by contrast, is a hypothesis that seems to work, but has not been shown to be *necessarily true*. To illustrate the difference between these two concepts consider the following proposition:

The sum of the first n odd numbers = n^2 (where n is any number).

If you are curious to know whether this is true, you might see what happens when you plug in the first odd number, then the first two odd numbers, then the first three odd numbers, and so on:

First	1	= 1	= 1^2	works
First two	1 + 3	= 4	= 2^2	works
First three	1 + 3 + 5	= 9	= 3^2	works
First four	1 + 3 + 5 + 7	= 16	= 4^2	works
First five	1 + 3 + 5 + 7 + 9	= 25	= 5^2	works

The proposition is true for every n we have tested. So can we say that we have *proved* that it is true – that it is true full stop? No! All we have done is reason **inductively**. You may remember that in Chapter 5 we said that induction involves reasoning from particular to general, and that although it is a useful way of reasoning it cannot give us certainty. No matter how many white swans you have seen, you cannot be sure that the next swan you see won't be black. Our claim about odd numbers works for the first five odd numbers; but that is no guarantee that it will work for the first twenty-four or hundred-and-four odd numbers. Relying on inductive reason we cannot be sure that at some point we won't encounter a metaphorical black swan.

To see the point, you might like to consider the following example. The question now is whether the following formula generates the sequence of square numbers (1, 4, 9, etc.) for any n.

$$n^2 + n \times (n - 1) \times (n - 2) \times (n - 3) \times (n - 4)$$

So let's test it:

When $n = 1$, we get: $1 + 1 \times (0) \times (-1) \times (-2) \times (-3) = 1$

When $n = 2$, we get: $4 + 2 \times (1) \times (0) \times (-1) \times (-2) = 4$

When $n = 3$, we get: $9 + 3 \times (2) \times (1) \times (0) \times (-1) = 9$

When $n = 4$, we get: $16 + 4 \times (3) \times (2) \times (1) \times (0) = 16$

Again, things seem to be working out. But now look at what happens, when $n = 5$:

When $n = 5$, we get: $25 + 5 \times (4) \times (3) \times (2) \times (1) = 145$

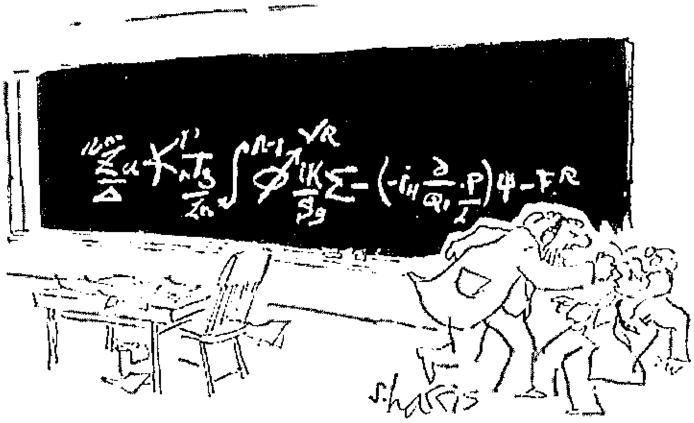
Shucks – a black swan! If you try it for 6, you get 756, and if you try it for 7, you get 2,569. In fact, beyond $n = 4$, the formula *never* generates the corresponding square number. (The example is in fact a contrived one. The formula was made in such a way that for numbers up to 4, one of the parts in brackets will sum to zero, thereby cancelling out everything to the right of the n^2. Beyond 4, this does not happen. Nevertheless, the example illustrates the danger of jumping to conclusions in mathematics.)

Now consider **Goldbach's conjecture** – a famous mathematical conjecture according to which every even number is the sum of two primes. If you try it out, it seems to work:

2 = 1 + 1	12 = 7 + 5
4 = 2 + 2	14 = 7 + 7
6 = 3 + 3	16 = 13 + 3
8 = 5 + 3	18 = 13 + 5
10 = 5 + 5	20 = 17 + 3

If you keep running through the even numbers into the hundreds, and the thousands and the tens of thousands, Goldbach's conjecture still works. So how far do you have to go before you can say you have *proved* it? Mathematicians have used computers to help test the conjecture for all the even numbers up to 100,000,000,000,000 and they have not found any counter-examples. You might think that is a good enough proof. But it is still abstractly possible that the next number – say, 100,000,000,000,002 – will not be the sum of two primes.

Furthermore, it is worth keeping in mind that although 100,000,000,000,000 may seem like a big number, compared with infinity, it's peanuts. For even a very large number is infinitely far away from infinity. The philosopher Ludwig Wittgenstein (1889–1951) put it well when he said 'Where the nonsense starts is with our habit of thinking of a large number as closer to infinity than a small one.' We may have tested Goldbach's conjecture up to 100,000,000,000,000, but the ratio of tested to untested cases is still incredibly small. (Imagine someone testing ten swans out of a total swan population of millions, and declaring on that basis that all swans are white.)



You want proof? I'll give you proof!

Figure 7.4

Most mathematicians believe that Goldbach's conjecture is in fact true, but since no one has yet shown that it is necessarily true for any randomly chosen even number, it remains one of the great unproven conjectures in number theory.

Before moving on, it is worth mentioning that the proposition on page 192, 'The sum of the first n odd numbers = n^2' has in fact been proved. We will not give a formal proof here but you can get a visual sense of it from Figure 7.5. If you begin from the top right square and then add successive odd numbers of small squares, first 3 squares, then 5 squares, then 7 squares, then 9 squares etc – you can see that each time they can be added to the previous square to make a larger one the side of which is equal to the number of odd numbers in the sequence.

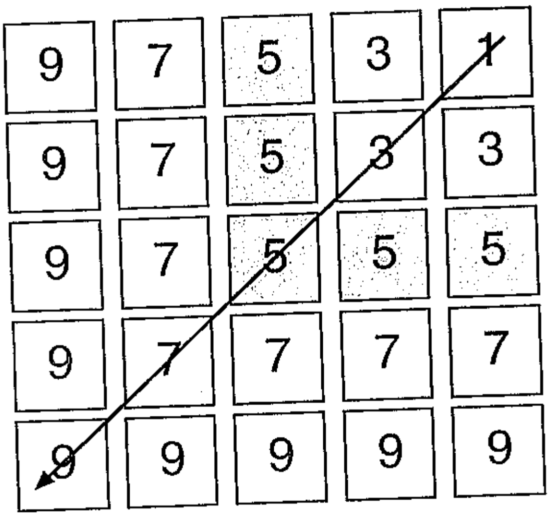


Figure 7.5

Beauty, elegance and intuition

Since any logical sequence of statements which leads to a theorem counts as a proof in mathematics, there may be many different proofs of a theorem. However, mathematicians generally seek proofs that are clear, economical and elegant. A particularly elegant proof might even be described as 'beautiful'. The great Hungarian mathematician, Paul Erdos (1913–96), used to speak of 'the BOOK' in which God keeps the most beautiful proofs for theorems; and he once joked that, even if God does not exist, you cannot doubt the existence of 'the book'.

Although the person in the street does not usually associate mathematics with beauty, we can get a sense of what mathematicians mean by a 'beautiful' or 'elegant' solution by considering a couple of simple examples.

1 There are 1,024 people in a knock-out tennis tournament. What is the total number of games that must be played before a champion can be declared?
2 What is the sum of the integers from 1 to 100?

The tennis tournament problem seems fairly straightforward, and to solve it you might reason as follows. In the first round, there will be 512 games, in the second round, 256, in the third round, 128, in the fourth 64, in the fifth 32, and then 16, 8, 4, 2, plus the final. Summing these figures, you get 1,023. But there is another simpler way of solving the problem. If 1,024 people enter the tournament, there will only be one person who wins all of their games, and that is the eventual winner. All of the other 1,023 players will lose one and only one game – for as soon as you have lost a game, you are out of the competition. Since every game results in one winner and one loser, there is a one-to-one correspondence between losers and games played. Therefore, there must have been 1,023 games. This explanation may sound a bit wordy when it is written out, but it is far more elegant and insightful than the standard way of solving the problem. While the first approach is, in effect, focusing on *winners* – there are 512 winners in the first round, 256 in the second round etc – the insight of the second approach is to change perspective and focus instead on *losers*.

Turning to the second problem, the standard approach of grinding through the arithmetic will – if you avoid careless errors – give you the right solution. 1 + 2 = 3, 3 + 3 = 6, 6 + 4 = 10, etc. However, there is, again, a more elegant way of solving the problem. If you write the numbers from 1 to 50 out from left to right, and then underneath the numbers from 51 to 100, but this time from right to left, you get the following:

1 2 3 4 5...	46 47 48 49 50
100 99 98 97 96	...55 54 53 52 51

When you lay the numbers out like this, you can see that if you sum each pair of numbers vertically, 1 + 100, 2 + 99,..., 49 + 52, 50 + 51, each pair sums to 101. How many pairs of numbers are there? 50! So the sum of the first 100 integers is

$50 \times 101 = 5{,}050$. This may again be clumsy to explain in words, but with this insight you can solve the problem in seconds rather than minutes.

The other interesting thing about the insightful solutions to the above two problems is that they can be *generalised*. We can say that in any knock-out tennis tournament of n entrants the total number of games played will be $n - 1$. And we can say that the sum of the first n integers is $\frac{1}{2}n(n + 1)$.

> Give a proof that the sum of the first n integers is $\frac{1}{2}n(n + 1)$.

What comes out of this discussion is that creative imagination and intuition play a key role in mathematics. When the German mathematician, David Hilbert (1862–1943) was told that one of his students had given up mathematics to become a novelist, he is said to have replied: 'It's just as well – he had no imagination.'

Although some mathematicians, such as Henri Poincaré (1854–1912), stress the role played by intuition in creative mathematical work, it is important to keep in mind the distinction we made in Chapter 6 between *natural intuitions* and *educated intuitions*.

> If you tie a string tightly around the 'equator' of a football, and you then want to add enough string to make it go all the way round the ball one inch from its surface – as in the diagram – it turns out that you will need to add about 6 inches to your original piece of string.
>
> Imagine that you tie a string going round the equator of the earth. (Assume the earth is a smooth sphere.) Again you decide you want the string to go round the earth one inch from its surface. How much will you need to add to the original length of string?
>
> **a** Make an intelligent guess about how much string you would need to add.
>
> **b** *Calculate* how much string you would need to add. (Hint π = circumference/diameter.)

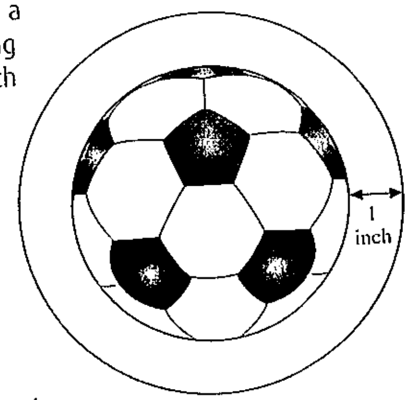


Figure 7.6

I imagine you were surprised to discover that the answer for the earth is the same as the answer for the football – roughly 6 inches. This goes against most people's natural intuitions. But a mathematician with educated intuitions might not be surprised by the result. However, no matter how good a person's mathematical intuitions are, such intuitions will not be accepted by the mathematical community until they have been proved.

1. Do you think that mathematical insight can be taught, or would you say that it is something inborn and either you've got it or you haven't?
2. We sometimes use calculators and computers to help us solve mathematical problems. Does it follow that machines *understand* mathematics?

Figure 7.7

Mathematics and certainty

Having said something about the nature of formal systems, we must now look in more detail at the nature of mathematical certainty.

To do this, let us begin by making two distinctions. The first concerns the nature of propositions. As we saw in the appendix to Part 2, an **analytic** proposition is one that is true by definition. We now add that a **synthetic** proposition is any proposition that is not analytic. So we can say that every proposition is either analytic or synthetic.

The second distinction concerns how we come to *know* that a proposition is true. A proposition is said to be knowable ***a priori*** if it can be known to be true independent of experience; and it is said to be knowable ***a posteriori*** if it cannot be known to be true independent of experience. As with the analytic–synthetic distinction, we can say that every true proposition can be known either *a priori* or *a posteriori*.

Combining the two pairs of distinctions, we can generate the following matrix:

		Nature of proposition	
		Analytic	Synthetic
How is it knowable?	*A priori*	1 ✓	4 ?
	A posteriori	2 ✗	3 ✓

Now let us try to explain what might fit into each of the four boxes:

Box 1 This concerns propositions that are true by definition and can be known independent of experience. Does anything go in this box? Yes! We can put all definitions in this box because they can all be known to be true independent of experience. You may recall the example on page 174 about Bolivian bachelors. I have never been to Bolivia and I know nothing about the profile of the average Bolivian bachelor, but I can say with complete confidence that every Bolivian bachelor is unmarried. Apart from a knowledge of the English language, I do not need any experience of the world to verify the truth of this proposition. It might, however, be described as a *trivial* truth. If you are told that all Bolivian bachelors are unmarried, you have learned nothing new about the world.

Box 2 For a proposition to go in box 2, it would have to be true by definition, but knowable only on the basis of experience. Now, if a proposition is true by definition, then we can know that it is true independent of experience. So the idea of an analytic *a posteriori* proposition is self-contradictory and box 2 is empty.

Box 3 This concerns propositions that are not true by definition and that cannot be known to be true independent of experience. Does anything go in this box? Yes – our **empirical** knowledge of the world! For example, the proposition 'There are elephants in Africa' is not true by definition, and its truth can be established only on the basis of experience – *a posteriori*. Someone actually has to go to Africa and see that there are some elephants there.

Box 4 What would something that goes in box 4 look like? Well, it must be a *non-trivial* proposition – i.e. one that is not true by definition – whose truth can be known independent of experience. Does anything go in this box? That is the million dollar question!

The question is: in which box should we put mathematics? Given that box 2 is empty, there seem to be three options.

> Look over the above terminology and then try to decide which of the available boxes you would put mathematics in and why.

Option 1: Mathematics as empirical

Some people, such as the philosopher John Stuart Mill (1806–72), have claimed that mathematics goes in to box 3. According to Mill, mathematical truths are empirical generalisations based on a vast number of experiences that are no different in kind from scientific statements such as 'All metals expand when heated.' Mill said that the reason we feel more certain that 2 + 2 = 4 than that all metals expand when heated is that we have seen so many more confirming instances of the former than of the latter, and this convinces us that 2 + 2 = 4 must be true.

1 Imagine there are two hungry lions in a cage. You open the cage door and throw in two lambs. How many animals are in the cage when you return the next day? Does this do anything to convince you that 2 + 2 is not always equal to 4?
2 Can you imagine a world in which 2 + 2 = 5? For example, what if every time you brought two pairs of objects close to one another, a fifth one popped into existence?
3 Does the fact that we usually teach children arithmetic by beginning with concrete objects, such as two apples and two apples, mean that arithmetic is an empirical subject?

Looking over the above questions, I doubt that the example of the lions and the lambs shakes your confidence in arithmetic. The fact that when you put two lions and two lambs together in a cage you do not end up with four animals but with two somewhat fatter lions tells you something about zoology, not arithmetic. The relevant arithmetical description of the situation would be $(2 + 2) - 2 = 2$.

I think that most people would deal with the second question in a similar way. There is clearly something weird about the physics of a world where a fifth object magically appears every time you bring two pairs of objects close to one another. But I don't think we would allow this fact to stand against the truths of arithmetic. We would probably say that in this world $2 + 2 + 1 = 5$. (Admittedly, the inhabitants of such a strange world might never develop arithmetic – but that is another issue.) What comes out of this example is that, while we can imagine a world in which the laws of physics are different, it seems to be impossible to imagine a world in which the laws of arithmetic are different.

What, if anything, can we infer from the fact that when we teach children arithmetic we usually begin with concrete objects? Let's take a closer look at the process. You show a child 2 apples and another 2 apples and ask, 'How many apples are there altogether?' She says '4'. You then do the same with oranges and bananas, and she comes up with the right answer each time. Then, at a certain point – and this is the crucial step – you ask the child, 'So what is 2 + 2?', and you hope that she makes the *leap of abstraction* and says '4'. What the child has to grasp is that she is not simply learning interesting facts about fruit and vegetables. She has to catch on to the idea that $2n + 2n = 4n$ for any n – even if she is not yet capable of expressing it that way herself. Once the child has 'got it', she will know that if a person has 2 aardvarks and buys 2 more aardvarks, he will end up with 4 aardvarks – even if she has no idea what an aardvark is. So, while the child's knowledge undoubtedly begins with experience, it ends up going beyond experience.

Imagine that you try to teach a child arithmetic by beginning with concrete examples in the way described above. When you present them with various quantities of apples or oranges, they can do the relevant sums, but they never make the 'leap of abstraction'. They accept that 2 apples + 2 apples = 4 apples in this case, but they keep insisting that they cannot see why this should always be true. What, if anything, could you do to convince them of the general truth that 2 + 2 = 4?

Option 2: Mathematics as analytic

The above kind of discussion is enough to convince many people that mathematics is not empirical. We might conclude from this that it must therefore be analytic and fit in box 1 in the above diagram. According to this view, if you understand the meaning of the terms in the proposition 2 + 2 = 4, you will see that it is true by definition. To say that 2 + 2 = 4 is essentially the same as saying (1 + 1) + (1 + 1) = (1 + 1 + 1 + 1). So when you solve a maths problem, you are simply unpacking a truth that is, in some sense, already contained in the statement of the problem. What is 2 + 2? The answer is already there – you just need to take the wrapping off!

Despite its plausibility, there are some problems with the idea that the whole of mathematics simply consists of strings of definitional truths. To start with, if it is all just true by definition, you might wonder why mathematics is so hard. One response might be that it is difficult to keep in mind long chains of reasoning when you are trying to solve complex problems, and it is therefore easy to make an error. So perhaps a precondition for being good at mathematics is having a good short-term memory.

Another problem with the analytic claim is that mathematical truths do not seem to be trivially true in the way that 'All bachelors are unmarried men' is trivially true. Take, for example, the fact mentioned above that the sum of the first n odd numbers equals n^2. This seems more like an interesting *discovery* about numbers than something that is true by definition. The claim that mathematics is analytic seems even less plausible when we consider **Goldbach's conjecture** which has not yet been shown to be true or false. For if mathematics is analytic, we now find ourselves in the strange position of having to say that if it is true that every even number is the sum of two primes, then it is true by definition; and if it is false that every even number is the sum of two primes, then it is false by definition. This implies that we do not yet know the proper definition of terms such as 'even number' and 'prime number'. Against this, it would surely be better to say that we know what even numbers are, but have not yet discovered all of their properties, just as we might say that we know what gold is, but have not yet discovered all of its properties.

A final problem with the analytic claim is that, if we say that mathematical propositions are true by definition, we are left with the puzzling fact that they seem to fit the world so well.

Option 3: Mathematics as synthetic *a priori*

A third option is to say that mathematics is neither empirical nor analytic, but *synthetic* a priori *knowledge* and goes in box 4. The suggestion now is that mathematics gives us non-trivial, substantial knowledge about the most general features of reality, and that this knowledge can be known to be true independent of experience. If this is true, then it means that, on the basis of reason alone, human beings are able to discover truths about the nature of reality.

This is pretty much how people thought about mathematics from the time of Euclid until the nineteenth century. The system of geometry that Euclid devised seemed to combine two features that were greatly valued by those who sought the truth. First, it seemed to be absolutely certain that, if you begin with self-evident axioms and use deductive reason, you arrive at true theorems. Second, it seemed to give substantial knowledge about the nature of physical space. After all, we can use geometry to divide up areas of land, build pyramids and estimate the circumference of the earth.

Not surprisingly, Euclidean geometry was seen by many as a model for the whole of knowledge, and it became the dream of many philosophers to do for knowledge in general what Euclid had done for geometry in particular. Thus the French philosopher René Descartes (1596–1650) sought to establish a system of philosophy founded on the self-evident first principle, *Cogito, ergo sum* – 'I think, therefore I am.' (Another philosopher called Baruch Spinoza (1632–77) wrote a whole book on ethics in which he tried to prove various theorems from what he believed to be ethical axioms.) Descartes, who has been called the father of modern philosophy, famously observed:

> To speak freely, I am convinced that it [mathematics] is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency, as being the source of all others.

1. To what extent do you think the geometric paradigm can be applied to other areas of knowledge?
2. What are the dangers of trying to extend geometrical thinking to other areas of knowledge?

If we decide that mathematical knowledge is indeed *synthetic a priori*, we are faced with the question of how the human mind is able to discover truth about the world on the basis of reason alone. One answer is to say that God created a 'pre-established harmony' between the human mind and the universe. This may be fine if you believe in God, but it is less convincing if you do not. An alternative is to say natural selection ensured that we evolved in such a way that our minds are in harmony with the environment. However, it is hard to see why mathematical ability would have given our remote ancestors an evolutionary advantage. A caveman didn't need calculus, and nature had no way of knowing that this would *eventually* turn out to be useful to the species. Perhaps, then, mathematical ability is a *by-product* of other abilities which do have survival value.

We have now considered three different views about the status of mathematical knowledge, and it may be worth putting a label on each of them.

Empiricism (Option 1)	box 3	Mathematical truths are empirical generalisations
Formalism (Option 2)	box 1	Mathematical truths are true by definition
Platonism (Option 3)	box 4	Mathematical truths give us *a priori* insight into the structure of reality. (This view can be traced back to Plato – hence the name.)

The empiricist view of mathematics has probably been the least popular of the three options, but you can find plenty of formalists and Platonists hiding in mathematics departments!

Discovered or invented?

A good way of highlighting the difference between Platonism and formalism is to consider the question, *Is mathematics discovered or invented?* While Platonists believe that mathematical entities are discovered and exist 'out there', formalists argue that they are invented and exist only 'in the mind'.

1 What is the difference between saying that something has been 'discovered' and saying that it has been 'invented'? What sorts of things do we usually say are discovered, and what sorts of things invented?

2 Do you think that intelligent aliens would come up with the same mathematics as us, or might they develop a completely different kind of mathematics?

At first sight, neither the 'discovered' nor the 'invented' option looks very attractive. If mathematical entities exist out there, does that mean that if we travel far enough in a space ship, we will one day encounter pi? And if mathematics is all in the mind, does that mean that mathematicians are simply making it up as they go along?

If we think more deeply about this issue, we run into puzzling questions about the meaning of the word 'existence'. We want to say that mathematical objects exist because we are able to make objective discoveries about them. You can, for example, prove that a circle encloses the largest possible two-dimensional area for a given perimeter, and anyone who follows your proof will come to the same conclusion. However, it turns out that mathematical objects, such as circles, do not exist in the real world. Wait a minute! Aren't coins and car wheels circles, and can't you draw a circle on a piece of paper any time you feel like it? In a strict mathematical sense, the answer is 'no'.

To see this, we need to go back to the definition of a circle – 'the set of all points in a plane that are equidistant from a given point'. With that definition in mind, try drawing a circle. If you do it free-hand, your drawing will be far from perfect. You will get a better result if you use a pencil and compass; but if you look at it with a magnifying glass, you will see that the border is fuzzy and it is not a perfect circle. You cannot solve the problem by using a finer pencil because if you increase the power of your magnifying glass the same fuzziness will appear again. Extrapolating from this, you can see that it is in fact impossible to draw a mathematically perfect circle – or any other geometrical object. A line, for example, is defined as that which has length but no breadth – and that is clearly something you cannot draw on a piece of paper. In a similar way, it turns out that there are no exact measurements either. For example, there is no such thing as exactly 4 centimetres – only exactly 4 centimetres 'to a given number of significant places'.

Such mathematical entities are **idealisations**; and, while you may get closer to the ideal, you will never be able to coincide with them.

The above discussion would seem to leave us with the following dilemma. On the one hand, since mathematical objects do not exist in the real world, they *must be mental fictions*. On the other hand, since we are capable of making discoveries about them, they *cannot be mental fictions*.

Plato's solution to the above dilemma was to say that mathematical objects exist 'out there', but not in the we-might-one-day-encounter-them-in-a-space-ship sense in which physical objects exist. Rather, they have their own unique way of existing; and although they cannot be perceived, they are just as real as physical objects. Indeed, Plato believed that they are *more real* than physical objects. How could anyone believe *that*?

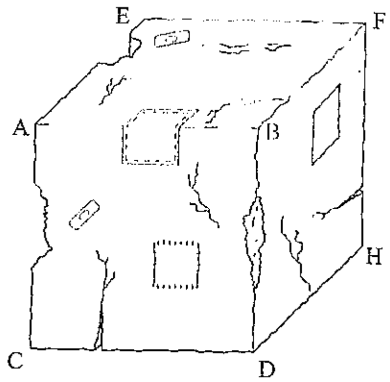


Figure 7.8 Platonism: physical reality is very different from mathematical perfection

Here's an updated Platonic argument. Consider the physical world. A physicist will tell you that the underlying reality of an everyday object, such as a table, is quite different from its appearance. Despite its apparent sturdiness and immobility, a table consists mainly of empty space in which atoms are whizzing around at great speed. Furthermore, tables and chairs – and human beings – come into being, exist briefly, and then return to dust. By contrast, mathematical objects have a clarity and immutability which – for Plato at least – gives them a superior existence. When you study Pythagoras' theorem at school, you are studying the same eternal truth that Pythagoras discovered more than two and a half thousand years ago; and it will be as true in a million years' time as it is now.

Plato's argument for the superior reality of mathematical over physical objects can be reduced to two key claims:

1 Mathematics is more certain than perception.
2 Mathematics is timelessly true.

1 'In order for something to exist, it must be possible to observe it.' Do you agree or disagree with this statement? Give reasons.

2 Do you think that numbers have always existed? Did they exist at the time of the Big Bang? If they exist, where do you suppose they exist?

3 Do you think that the full expansion of pi, which goes on for ever, exists 'out there', and that we are gradually discovering more and more about it?

Criticisms of Platonism

While Platonism continues to be popular with some mathematicians, others dismiss it as 'pi in the sky' mysticism. Two main objections can be made against it:

1. Since the series of natural numbers is infinite, Platonism is committed to the view that there are literally an infinite number of abstract mathematical entities 'out there'. This seems hard to believe. If we abandon the idea that reality consists only of observable entities, we seem to be in danger of wandering off into mysticism.
2. If mathematical objects have some weird kind of ideal existence, then how can physically embodied beings such as ourselves get to know about them?

According to formalism, which sees the above objections as decisive, mathematics consists of nothing but man-made definitions, axioms and theorems. We might liken it to a game of chess. A certain position on a chessboard is like a theorem that follows from the 'axioms of chess'. Since no one would say that all possible chess moves exist 'out there' in a Platonic heaven, why should we be tempted to say this about mathematics?

Some philosophers have suggested that rather than think of mathematical entities as being either objective or subjective, we should think of them as having 'social existence'. Consider again the game of chess. In what sense does it exist? Does it still exist if no one is playing chess? What about if no one is thinking about chess? Would it still exist if we destroyed all the rule books on how to play chess? Most people would probably say that even if no one played chess for a year, there would still be certain statements about it that are true and others that are false. However, if the human race disappeared it wouldn't make much sense to say that the game of chess still existed.

> Although Romeo and Juliet are fictional characters, it is true to say that Romeo loves Juliet and false to say that he hates Juliet. In what ways are mathematical objects similar to fictional characters and in what ways are they different?

Well, enough of these vexing questions about the meaning of existence. To develop our discussion about the nature of mathematics further, we must now look at the rise of alternative – 'non-Euclidean' – systems of geometry in the nineteenth century.

Non-Euclidean geometry and the problem of consistency

As we said earlier, Euclidean geometry was for many centuries seen as a model of knowledge because it seemed to be both certain and informative. There was, however, one small problem. The certainty of geometry was supposed to be guaranteed by the fact that one began with self-evident axioms and used deductive reason to derive theorems. However, one of Euclid's axioms, the axiom of parallels – which says that there is just one straight line through a given point which is parallel to a given line – struck people as being less self-evident than the other axioms. This doubt may have arisen from the fact that parallel lines are by definition lines that never meet even if you extend them to infinity – but who is to say what happens at infinity? Since mathematicians wished to get rid of all possible doubt, they expended a great deal of energy over the centuries in trying to demonstrate that the axiom of parallels was in fact a theorem. But no one succeeded in doing this!

Riemannian geometry

Then in the nineteenth century, a mathematician called Georg Friedrich Bernard Riemann (1822–66) came up with the clever idea of replacing some of Euclid's axioms with their contraries. Most people thought that if you based a system of geometry on non-Euclidean axioms, the system would lead to a contradiction and so collapse. This would then show that Euclid's axioms were in fact the only possible ones. However, to people's amazement, no contradictions turned up in Riemann's system.

Riemann's axioms differed from Euclid's as follows:

A Two points may determine more than one line (instead of axiom 1).

B All lines are finite in length but endless – i.e. circles (instead of axiom 2).

C There are no parallel lines (instead of axiom 5).

Among the theorems that can be deduced from these axioms are:

1. All perpendiculars to a straight line meet at one point.
2. Two straight lines enclose an area.
3. The sum of the angles of any triangle is greater than 180 degrees.

These theorems sound pretty strange. How can perpendiculars possibly meet at a point, or two straight lines enclose an area, or the angles of a triangle sum to more than 180 degrees? Fortunately, we can give intuitive sense to Riemannian geometry by imagining that space is like the surface of a sphere. Since we live on the surface of a sphere (more or less), this should not be too difficult to do!

The key to making sense of Riemann's system is to think about what a straight line will look like on the surface of a sphere. What is a straight line? The shortest distance between two points! Now, on the surface of a sphere, it can be shown that the shortest distance between two points is always an arc of a circle whose centre is the centre of the sphere. Such 'great circles' include not only all lines of longitude, but an endless number of other circles – as can be seen from the following diagram. (The only line of latitude that is a great circle is the equator.)

What this means is that, in Riemannian geometry, a straight line will appear curved when it is represented on a two-dimensional map. To illustrate this point, look at any airline flight map. Although the flight paths look curved, since airlines are in the business of making money, you can be sure that in reality they always take the shortest route to their destination.

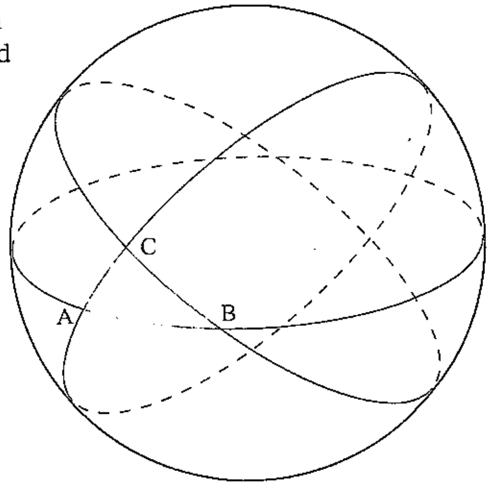


Figure 7.9 Riemannian geometry

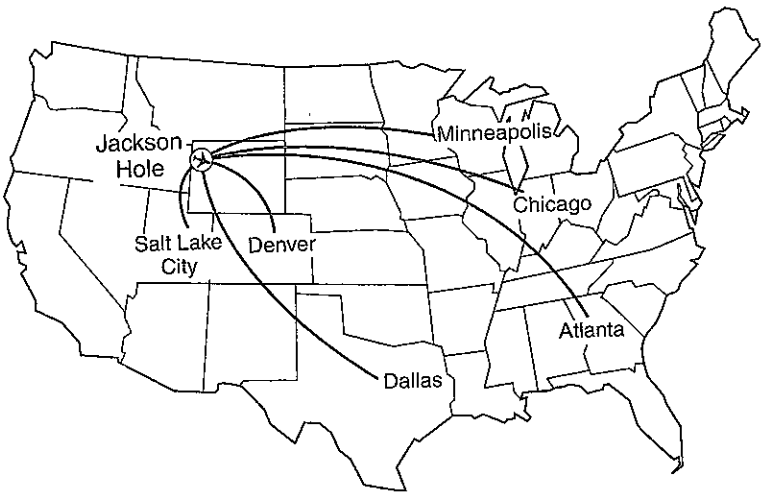


Figure 7.10 Flight paths

Once we have clarified the meaning of a straight line in Riemannian geometry, we can give a meaning to the three theorems mentioned above:

1. *All perpendiculars to a straight line meet at one point* Lines of longitude are perpendicular to the equator, but they meet at the north pole.
2. *Two straight lines enclose an area* Any two lines of longitude (straight lines) meet at both the north and south pole and so define an area (see Figure 7.9).
3. *The sum of the angles of any triangle is greater than 180 degrees* This can be seen in Figure 7.9.

With our discussion of Riemannian geometry in mind, try to solve the following puzzle. A hunter leaves his house one morning and walks one mile due south. He then walks one mile due west and shoots a bear, before walking a mile due north back to his house. What colour is the bear?

The problem of consistency

Although Riemann did not find any contradictions in his system of geometry, some of his contemporaries were convinced that sooner or later a contradiction would be found and Riemann's system would collapse. After all, he had not *proved* that his system was free from contradiction. As a result of this, mathematicians became increasingly interested in the problem of consistency, and the question of how we can be sure that any given formal system is free from contradiction.

You might think that if no contradictions have been found in a formal system that has been studied by the mathematical community then that is good enough. After all, people have been using Euclidean geometry for thousands of years, and not a single contradiction has turned up. While this would be enough to convince most people, it is not mathematically compelling. For it is still only a *conjecture* and it does not *prove* that the system in question is free from contradiction.

Another approach might be to appeal to intuition. Surely, if you begin with things that are intuitively obvious and reason consistently, you can be confident you will not run into contradictions. The problem with this is that, as we saw in Chapter 6, our intuitions can sometimes let us down.

To illustrate, consider the following story of a barber who had an affair with the king's daughter. When the king discovered what had happened, he was very angry and wanted to put the barber to death. But his daughter begged him to spare the barber's life. Wishing to appear merciful while ensuring that the barber eventually died, the king came up with a cunning plan. He told the barber that he would not execute him if he obeyed one simple instruction. He was to go back to his village the following day and *shave all and only those inhabitants who do not shave themselves*. 'Wow', thought the barber, 'that's easy!' Overjoyed, he headed back to his village and the next day he got to work shaving all and only those inhabitants who did not shave themselves. By dusk, he had completed his task, and tired but happy he returned home. Opening the front door, he happened to glance in the mirror in the hallway. 'Ah', he thought, 'I've missed someone. I, too, am an inhabitant of this village; so the king's instruction also applies to me.' As the barber turned this over in his mind, it slowly dawned on him that he had been trapped. For according to the king's instruction, if he shaved himself, then he shouldn't shave himself, and if he didn't, then he should! The instruction which had sounded simple enough turned out to be impossible to fulfil, and the barber was duly executed. Given our interests, the moral of the tale is not 'Don't mess with kings' daughters' but 'Apparently clear instructions can sometimes be impossible to fulfil.' What this means in terms of mathematics is that intuition alone is no guarantee that a system is free from contradiction.

Gödel's incompleteness theorem

The concern with consistency continued into the twentieth century. Finally in 1931, a young Austrian mathematician called Kurt Gödel (1906–78) came up with an extraordinary proof, known as ***Gödel's incompleteness theorem***, which shook the mathematical world to its foundations. What Gödel proved was that *it is impossible to prove that a formal mathematical system is free from contradiction*. We need to be a little careful here: Gödel did not prove that mathematics actually contains contradictions, but that we cannot be certain that it doesn't. What this means is that, at an abstract level, even mathematics is unable to give us certainty. For it is always possible that one day we will find a contradiction; and one small contradiction in a formal system would be enough to destroy the entire system. A mathematician friend of mine said that the first time he read Gödel's theorem it made him 'very sad'. Since as a matter of fact no contradictions have ever been discovered in the structure of mathematics, most mathematicians do not lose any sleep over Gödel's theorem, and they take a 'business as usual' approach to the subject. Nevertheless, there is a sense in which, after nearly two and a half thousand years, the last bastion of certainty has been breached by the turbulent waters of doubt.

Our story ends with not only the *certainty*, but also the *informative content* of Euclidean geometry coming under fire. For if there are alternative systems of geometry, the question now arises: 'Which one provides the best description of physical reality?' Although Euclidean geometry clearly works well enough at the 'local' level, according to Einstein's theory of relativity, it turns out that the universe obeys the rules not of Euclidean but of Riemannian geometry. According to our best scientific theories, space is curved!

Applied mathematics

To complete our discussion of mathematics, we now turn to applied mathematics – mathematics that is used to model and solve problems in the real world. Our discussion in the last section has shown that we can no longer unquestioningly assume that human reason gives us insight into the structure of reality. For the main alleged example of such rational insight – Euclidean geometry – turned out to be a false description of reality. At the same time, mathematics is still amazingly useful, and it is hard to avoid Galileo's conviction that the book of nature is written in the language of mathematics.

A particularly mysterious feature of the relationship between mathematics and the world is that mathematical ideas that are developed as a purely intellectual exercise sometimes turn out to be applicable to the real world. For example, in the third century BCE, the Greeks became interested in the geometry of **ellipses**, and a mathematician called Apollonius of Perga (*c.* 262–*c.* 190 BCE) wrote eight weighty volumes about them. As fascinating as the topic was – at least to Apollonius – the knowledge he acquired was completely useless. If someone had asked him what the point of studying ellipses was, he might have quoted Euclid's crushing response to a student who asked a similar question: 'Give him a penny – he wants to profit from his learning!' The pursuit of such knowledge was considered to be an end in itself that did not need a practical justification. The strange thing is that, when the seventeenth-century astronomer Johannes Kepler (1571–1630) was studying planetary motion, he discovered that, rather than being circular, as he had previously believed, the orbits of the planets round the sun are in fact elliptical. After being of merely academic interest for nearly two thousand years, Apollonius' work turned out to be of practical value!

Similarly, although Riemann developed non-Euclidean geometry as a purely intellectual exercise, thirty years later, Einstein concluded that space conforms to Riemannian rather than Euclidean geometry.

> To what extent do you think governments should fund 'useless' research in pure mathematics?

So how can we explain what one physicist has described as 'the unreasonable effectiveness of mathematics'? Here is what Einstein had to say on the subject:

> How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things? In my opinion, the answer to the question is briefly this: – As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

This is an interesting way of looking at the connection between mathematics and the world. What Einstein is saying is that mathematical systems are *invented*, but it is a matter of *discovery* which of the various systems apply to reality. You can invent any formal system you like and prove theorems from axioms with complete certainty. However, once you ask which system applies to the world, you are faced with an *empirical* question which can only be answered on the basis of observation. Thus Einstein discovered that Riemannian geometry is a better description of physical space than Euclidean geometry.

Now, you might ask why *any* purely invented system should have application to reality. A possible response is that some of the formal systems we invent are originally suggested to us by reality. For example, since geometry first arose in response to practical problems, and was then formalised by Euclid, it is perhaps not surprising that Euclidean geometry turned out to be a useful way of describing reality. The point, in other words, is that, even if mathematics is a game, the rules for the most interesting or useful games may be suggested to us by reality.

Nevertheless, there are many unexpected connections in mathematics that are difficult to explain. For example, π – which, as we saw earlier, is the circumference of a circle divided by its diameter – turns up in all kinds of quite unrelated places, such as the solution to **Buffon's needle problem**. This problem was posed by the French mathematician the Comte de Buffon (1707–78). Suppose you have a large sheet of paper ruled with parallel lines drawn at one unit intervals resting on a flat surface and you then throw a needle which is one unit long at random on to the paper. What is the probability that it will intersect one of the lines?

The answer to the problem turns out to be $2/\pi$. But why π should turn up in this context is a complete mystery.

At a deep level, then, there remains something mysterious about the 'unreasonable effectiveness of mathematics'. This is, I think, connected to the equally perplexing question of why there is order in the universe – in particular, order of the kind that can be uncovered by mathematical thinking. Perhaps all we can say is that if there wasn't any order we wouldn't be around to ask why not!

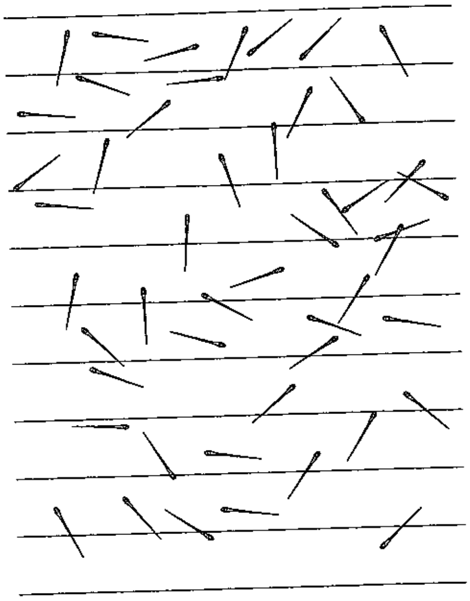


Figure 7.11 Buffon's needle problem

Conclusion

At the beginning of this chapter, we defined mathematics as 'the science of rigorous proof' and we spoke of the commonly held view that it is an island of certainty in an ocean of doubt. There is something immensely appealing about the idea of demonstrating something in such a way that any rational person will come to the same conclusion, and it is not surprising that mathematics has often served as a model for knowledge.

Nevertheless, we have seen that even in this most rigorous of subjects there are limits to certainty. At an abstract level, Gödel showed that we can never prove that mathematics is free from contradiction; and although this is unlikely to keep mathematicians awake at night, it means that the dream of absolute certainty will never be realised. At a more practical level, we have seen that when mathematics is applied to the real world we usually have a choice of axioms and we can only decide which are the most useful by testing them against reality.

Although mathematics cannot give us absolute certainty, it continues to play a key role in a wide variety of subjects ranging from physics to economics, and there is something surprising and mysterious about its extraordinary usefulness. Nevertheless, it is important to keep in mind that we cannot capture everything in the abstract map of mathematics and, despite its value, there is no reason to believe that it is the only, or always the best, tool for making sense of reality.

- Mathematics, which can be defined as 'the science of rigorous proof', begins with axioms and uses deductive reason to derive theorems.
- Although proof is the logical matter of deriving theorems from axioms, mathematicians consider some proofs to be more beautiful than others.
- According to three different views about the nature of mathematical truths they are either: (1) empirical, (2) true by definition, or (3) rational insights into universal truths.
- While some people believe that mathematics is *discovered,* others claim that it is *invented*; but neither view seems to be entirely satisfactory.
- The development of non-Euclidean geometries in the nineteenth century raised the question of how we can be sure that a formal system is free from contradiction.
- Kurt Gödel proved that it is impossible to prove that a formal mathematical system is free from contradiction.
- Mathematicians and philosophers are still perplexed by the extraordinary usefulness of mathematics.

Terms to remember

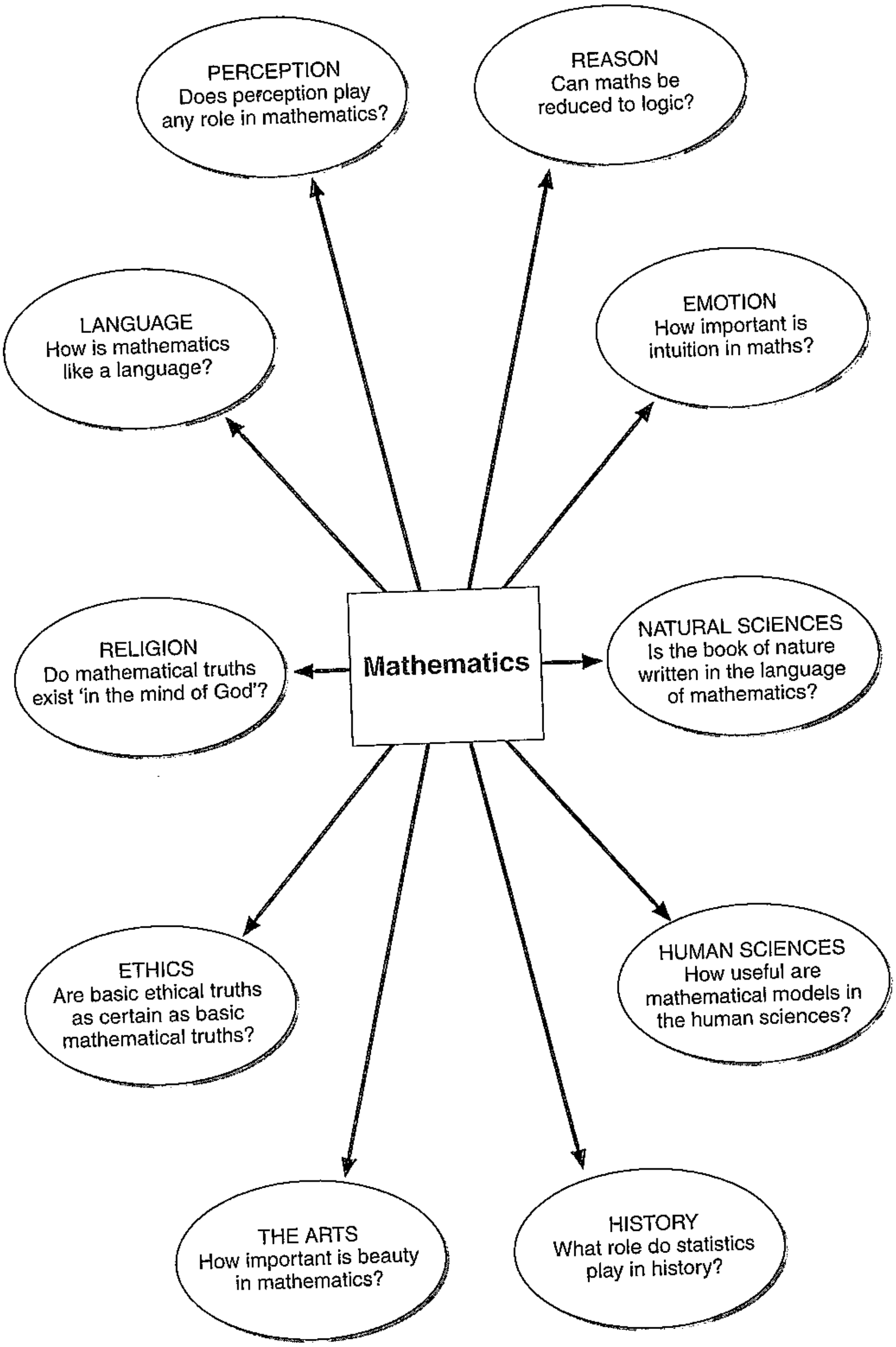
analytic
a posteriori
a priori
axioms
conjecture
deduction
empiricism
Euclidean geometry
formal system
formalism
Gödel's incompleteness theorem
Goldbach's conjecture
idealisation
Platonism
synthetic
theorem

Further reading

G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1994). In this short, readable book, G. H. Hardy seeks to justify his devotion to mathematics despite his insistence that '"real" mathematics is almost entirely "useless"'. The book contains many thought-provoking comments about the nature of mathematics and its relation to the arts and sciences. You don't have to be a mathematician to enjoy it.

John Allen Paulos, *Innumeracy: Mathematical Illiteracy and its Consequences* (Vintage, 1990). Paulos defines 'innumeracy' as 'an inability to deal comfortably with the fundamental notions of number and chance'. His book is full of entertaining examples of how we misinterpret data and misunderstand probability. The chapter on the relation between innumeracy and pseudo-science is particularly interesting.

Linking Questions



Reading Resources

WHY IS MATH SO USEFUL?

The universe appears to speak the language of mathematics. In what follows, Michael Lemonick considers the mystery of why mathematics is so useful.

My father is a physicist, and I learned early on, sitting at the dinner table while he talked with his colleagues, that physicists speak a different language from the rest of us. They would fill the air with such phrases as 'angular momentum' and 'virtual particle' and 'photon,' while I sat awed by the thought of the mysterious ideas their conversations concealed.

I didn't realize at the time how deep the mystery went. As I grew old enough to ask questions, I found out that these exotic terms aren't part of the true language of physics at all, but only approximations. The universe works in a way so far removed from what common sense would dictate that words of any kind must necessarily be inadequate to explain it. The only way to describe what really goes on, I was told, is to speak in mathematics.

I learned about photons, the smallest conceivable bit of light. Sometimes a photon behaves like a particle, sometimes like a wave, depending on how you look at it. When it's a wave, it isn't a wave of anything. It's just a wave. If it ever came to rest, a photon would have no mass, but since it always travels at the speed of light, it does have mass. This didn't make sense to me; the reason, I was told, was that I didn't understand the math.

The speed of light itself turned out to be an equally baffling phenomenon. Imagine two photons, each rushing away from the same light bulb in exactly opposite directions. How fast are they moving away from each other? My answer: twice the speed of light. The correct answer: half that amount, as calculated with the equations of general relativity, which are yet to make an inaccurate prediction.

It always seemed curious to me that mathematics, so thoroughly a non-experimental science, should be so powerfully descriptive of the natural world. For example, Greek mathematicians invented ellipses purely as an intellectual exercise; they are, quite literally, figments of human imagination. It was centuries before anyone realized the planets move in elliptical paths.

It turns out that some physicists find this relationship curious too. In 1960, Eugene Wigner, a Hungarian emigre who would win a 1963 Nobel prize for his work on quantum mechanics, published an essay in the journal *Communications on Pure and Applied Mathematics*. Entitled 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences,' it points out just how deep the mystery goes.

According to Wigner, some of the most important concepts in physics, including quantum theories and theories of gravitation, owe their success to mathematical systems devised without any idea they would someday be applied. 'It is difficult to avoid the impression that a miracle confronts us here,' he wrote.

Uncanny predictions

The first case he cites is Newton's law of gravitation, which states that the motion of a freely falling object – say, an apple – and the motions of planets, satellites and stars are special cases of the same phenomenon, describable by one set of equations. In this case, mathematician and physicist were the same person: He invented calculus, then applied it. (In the Greek tradition, Newton believed mathematics was too pure to be sullied by association with the real world. He wasn't entirely happy with his discovery.)

It happened again when physicists noticed similarities between the structure of quantum mechanics and a mathematical system called matrix theory. They made predictions based on the similarities, and the predictions were confirmed.

Other such serendipitous matchings have been noted as well. Writing in the October 1984 issue of the *American Journal of Physics*, William Pollard, of the Institute for Energy Analysis, in Oak Ridge, Tennessee, points out several: Einstein's equations of general relativity are based on the nineteenth-century, many-dimensional mathematics of Bernhard Riemann. The theory of quarks, the basic building blocks of matter, is based on a form of algebra concocted by a Norwegian mathematician, Sophus Lie, long before protons and neutrons were even postulated.

Can these all be coincidences? Neither Wigner nor Pollard thought so. Somehow the human mind seems to have a built-in capacity to deduce the structure of the universe without observing it first. It is nearly impossible to believe, and quite impossible to explain, but perhaps the physical laws governing the atoms in our brain tissues push our thinking in the direction of understanding those laws. As Wigner says, and Pollard repeats, 'The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.' The sense of mystery felt at those long-ago dinner-table discussions put me, it seems, in very good company.

THINK MATHS

Is mathematics the grand design for the Universe, or merely a figment of the human imagination, asks Ian Stewart.

Where does mathematics come from? Is it already out there, waiting for us to discover it, or do we make it all up as we go along? Plato held that mathematical concepts actually exist in some weird kind of ideal reality just off the edge of the Universe. A circle is not just an idea, it is an ideal. We imperfect creatures may aspire to that ideal, but we can never achieve it, if only because pencil points are too thick. But there are those who say that mathematics exists only in the mind of the beholder. It does not have any existence independent of human thought, any more than language, music or the rules of football do.

So who is right? Well, there is much that is attractive in the Platonist point of view. It's tempting to see our everyday world as a pale shadow of a more perfect, ordered, mathematically exact one. For one thing, mathematical patterns permeate all areas of science. Moreover, they have a universal feel to them, rather as though God thumbed His way through some kind of mathematical wallpaper catalogue when He was trying to work out how to decorate His Universe. Not only that: the deity's pattern catalogue is remarkably versatile, with the same patterns being used in many different guises. For example, the ripples on the surface of sand dunes are pretty much identical to the wave patterns in liquid crystals. Raindrops and planets are both spherical. Rainbows and ripples on a pond are circular. Honeycomb patterns are used by bees to store honey (and to pigeonhole grubs for safekeeping), and they can also be found in the geographical distribution of territorial fish, the frozen magma of the Giant's Causeway, and rock piles created by convection currents in shallow lakes. Spirals can be seen in water running out of a bath and in the Andromeda Galaxy. Frothy bubbles occur in a washing-up bowl and the arrangement of galaxies.

With this kind of ubiquitous occurrence of the same mathematical patterns, it is no wonder that physical scientists get carried away and declare them to lie at the very basis of space, time and matter. Eugene Wigner expressed surprise at the 'unreasonable effectiveness' of mathematics as a method for understanding the Universe. Many philosophers and scientists have seen mathematics as the basis of the Universe. Plato wrote that 'God ever geometrises'. The physicist James Jeans declared that God was a mathematician. Paul Dirac, one of the inventors of quantum mechanics, went further, opining that He was a pure mathematician. In the past few years Edward Fredkin has argued that the Universe is made from information, the raw material of mathematics.

This is powerful, heady stuff, and it is highly appealing to mathematicians. However, it is equally conceivable that all of this apparently fundamental mathematics is in the eye of the beholder, or more accurately, in the beholder's mind. We human beings do not experience the Universe raw, but through our senses, and we interpret the results using our minds. So to what extent are we mentally selecting particular kinds of experience and deeming them to be important, rather than picking up things that really are important in the workings of the Universe? Is mathematics invented or discovered?

If pushed, I would say that it is a bit of both because neither word adequately describes the process. Moreover, they are not alternatives, they are not opposites, and they do not exhaust the possibilities. They are not even particularly appropriate. We use 'discover' for finding things that already exist in the physical world. Columbus discovered America – it was already there, but neither he nor anyone else where he came from knew it was – and David Livingstone discovered the Victoria Falls. The word 'invention' means bringing into existence something that was not previously there. Edison invented electric light, Bell invented the telephone.

However, when Columbus landed in America he was actually trying to invent a new trade route to India. And Livingstone's discovery came as no great surprise to the local inhabitants, who saw the Victoria Falls every day. Edison would have felt as if he had invented the idea of electric lighting, but then spent many years trying to discover how to make it a reality. So invention and discovery both happen within a particular context – people becoming aware that there is something new in their world.

It is the same with mathematics. What to the outside world looks like invention often feels more like discovery to insiders. The distinction is made all the more tricky because mathematical objects lead a virtual existence, not a real one: they reside in minds, not embodied in any kind of hardware. But unlike, say, poetry, that virtual world obeys rigid rules, and those rules are pretty much the same in every mathematical mind.

In a way, the world of mathematical ideas is a kind of virtual collective comparable to Jung's famous 'collective unconscious' – the idea that all human minds have access to vast, evolutionarily ancient, subconscious structures and processes that govern much of our behaviour. But in what sense are they 'collective'? A crucial distinction has to be made here between a single unconscious entity, into which we all dip, and a large number of distinct but very similar unconsciousnesses, one for each of us. It is the difference between a community with a single municipal swimming pool, and one in which every back garden has its own pool.

From the point of view of specific action, the distinction is not terribly important: you can discuss the problems of keeping leaves out of 'the pool' with your neighbour without ever making it clear whether you think of it as a single common pool, or a typical representative of the individual pools that everybody has. But if you want to understand what's going on in general, then it does make a difference. The notion of a single unconscious mind for all of humanity is a mystical and rather silly concept that leads in the direction of telepathy. A collection of more or less identical individual subconsciousnesses, rendered similar by their common social context, is considerably more prosaic but a great deal more sensible.

The same point lies at the heart of how I think we should view mathematics. Because we have a single word for the virtual collective it is tempting to think of it as a single thing – like Jung's mystical telepathic unconscious–into which all mathematicians dip. This is a difficult concept to capture. Where is that thing? What is it made of? How does it grow? Instead, it is better to think of mathematics as being distributed throughout the minds of the world's mathematicians. Each has his or her own mathematics inside his or her head. Moreover, those individual systems are extremely similar to each other, much more so than Jungian subconsciousnesses. Not in the sense that each head contains the whole of mathematics. Mine contains dynamical systems, yours contains analysis, and hers algebra, say. But all three are logically consistent with each other, because of how mathematicians are trained, and how they communicate their ideas. If what is in my head is not consistent with what is in yours, then one of us has got it wrong and we will argue until it becomes clear to us both who it is.

Most areas of human activity are structured in this way. So the difficult questions of existence and discovery versus invention are not confined to mathematics. Take medicine, for example. What is medicine? Where does it live? Is it invented or discovered? Now replace medicine by plumbing, ballet, football, language or cycling, and it is clear just how widespread the structure is, and why the question doesn't make a great deal of sense in any area of human activity. What goes on is neither invention nor discovery, but a complex context-dependent mix of both.

When it comes to mathematics, sometimes it really does feel like discovery. When you are carrying out mathematical research in a previously defined area it feels like discovery because there is no choice about what the answer is. But when you are trying to formalise an elusive idea or find a new method, it feels more like invention: you are floundering around, trying all sorts of harebrained ideas, and you simply do not know where it will all lead. The more established an area of mathematics becomes, the more strongly it feels as if there is some kind of fixed logical landscape, which you merely explore. Once you've made a few assumptions (axioms), then everything that follows from them is predetermined. But this account misses out the most crucial features: significance, simplicity, elegance, how compelling the argument is, all things that give the landscape its character.

But if mathematics resides in mathematicians' heads, why is it so 'unreasonably effective'? (E. Wigner) The easy answer is that most mathematics starts in the real world. For instance, after observing on innumerable occasions that two sheep plus two more sheep make four sheep, ditto cows, wolves, warts and witches, it is a small step to introduce the idea that $2 + 2 = 4$ in a universal, abstract sense. Since the abstraction came out of reality, it's no surprise if it applies to reality.

However, that is too simple-minded a view. Mathematics has an internal structure of logical deduction that allows it to grow in unexpected ways. New ideas can be generated internally too, whenever anyone tries to fill obvious holes in the logical landscape. For example, having worked out how to solve quadratic equations, which arose from problems about baking bread, or whatever, it is obvious that you ought to try to solve cubic and quintic equations too. Before you can say 'Evariste Galois' you're doing Galois theory, which shows that you can't solve quintics, but is almost totally useless for anything practical. Then someone generalises Galois theory so that it applies to differential equations, and suddenly you find applications again, but to dynamics, not to bakery.

Yes, there is a flow of problems and concepts from the real world into mathematics, and a back-flow of solutions from mathematics to reality. Wigner's point is that the back-flow may not answer the problem that you set out to solve. Instead, it may answer something just as real, just as important, but physically unrelated. Why should this be? Well, mathematics is the art of drawing necessary conclusions, independently of interpretations. Two plus two has to be four, whether you are discussing sheep, cows or witches. In other words, the same abstract structure can have several interpretations. So you can get the ideas from one interpretation, and transfer the result to others. Mathematics is so powerful because it is an abstraction.

This is all very well, but why do the abstractions of mathematics match reality? Indeed, do they really match, or is it all an illusion? Enter cultural relativism – the idea that has lately become so fashionable in academic arts departments, which sees maths and science as social constructs no less and no more valid than any other social construct. Does this lead to the idea that science can be anything scientists want it to be?

True, science is a social construct. Scientists who claim that it is not are making the same mistake as those who think that we all dip into the same collective subconscious. But there is something special about science: it is a construct that has at every step been tested against external reality. If the world's scientists all got together and decided that elephants are weightless and rise into the air if they are not held down by ropes, it would still be foolish to stand under a cliff when a herd of elephants was leaping off the edge. In science, there has to be a reality check. Because it is done by beings who see reality through imperfect and biased senses, the reality check cannot be perfect, but science still has to survive some very stringent scrutiny.

So what's the reality check in maths? Well, the deeper we delve into the 'fundamental' nature of the Universe, the more mathematical it seems to get. The ghostly world of the quantum cannot be expressed without mathematics: if you try to describe it in everyday language, it makes no sense ...

Human minds evolved in the real world, and they learnt to detect patterns to help us survive events outside ourselves. If none of the patterns detected by these minds bore any genuine relation to the real world outside, they wouldn't have helped their owners survive, and would eventually have died out. So our figments must correspond, to some extent, to real patterns. In the same way, mathematics is our way of understanding certain features of nature. It is a construct of the human mind, but we are part of nature, made from the same kind of matter, existing in the same kinds of space and time as the rest of the Universe. So the figments in our heads are not arbitrary inventions. There are definitely some mathematical things in the Universe, the most obvious being the mind of a mathematician. Mathematical minds cannot evolve in an unmathematical universe. Only a geometer God can create beings able to come up with geometry.

But that is not to say that only one kind of mathematics is possible: the mathematics of the Universe. That seems too parochial a view. Would aliens necessarily come up with the same kind of mathematics as us? I don't mean in fine detail. For example the six-clawed cat creatures of Apellobetnees Gamma would no doubt use base-24 notation, but they would still agree that twenty-five is a perfect square, even if they write it as 11. However, I'm thinking more of the kind of mathematics that might be developed by the plasma vortex wizards of Cygnus, for whom everything is in constant flux. Maybe they'd understand plasma dynamic a lot better than we do, though I suspect we wouldn't have any idea how they did it. But I doubt that they would have anything like Pythagoras's theorem. [$H^2 = \sqrt{(a^2 + b^2)}$] There are few right angles in plasmas. In fact, I doubt they'd have the concept 'triangle'. By the time they had drawn the third vertex of a right triangle, the other two would be long gone, wafted away on the plasma winds.